ADDITIONAL EXERCISES FOR

MATHEMATICS 144 — PART 4

Fall 2017

V. Number systems and set theory

V.2. Finite induction and recursion

Additional problems

113. Let $\mathbb{Q}[t_1, \cdots, t_n]$ denote the set of all polynomials in the n indeterminates $t_1, \cdots, t_n$with rational coefficients. The **degree** of a nonzero polynomial $p \in \mathbb{Q}[t_1, \cdots, t_n]$ is defined (as usual) to be the largest n such that p has a summand which is a nonzero multiple of some monomial $t_1^{a(1)} \cdots t_n^{a(n)}$ such that $\sum a(k) = n$. A polynomial p of positive degree is said to be *irreducible* if it cannot be written as a product of two polynomials $q_1 q_2$ where both q_1 and q_2 have positive degree. Prove that every positive degree polynomial in $\mathbb{Q}[t_1, \cdots, t_n]$ is a product of irreducible polynomials. [*Note:* A result of C. F. Gauss implies that the irreducible factors are unique up to multiplication by a nonzero constant.]

114. Let $\mathbb{Z}[\sqrt{5}]$ be the set of all real numbers expressible as $a + b\sqrt{5}$ where a and b are integers. Define the *absolute norm* of $a + b\sqrt{5}$ to be $|N(a + b\sqrt{5}| = |a^2 - 5b^2|$. If $a + b\sqrt{5}$ is nonzero, then the irrationality of $\sqrt{5}$ implies that $0 < |N(a + b\sqrt{5}| \in \mathbb{Z}$.

(*i*) If $x, y \in \mathbb{Z}[\sqrt{5}]$ prove that $|N(x \cdot y)| = |N(x)| \cdot |N(y)|$.

(*ii*) If $x \in \mathbb{Z}[\sqrt{5}]$ prove that $x^{-1} \in \mathbb{Z}[\sqrt{5}]$ if and only if $|N(x)| = 1$.

(*iii*) An element of $\mathbb{Z}[\sqrt{5}]$ whose absolute norm is greater than 1 is said to be irreducible if it cannot be written as a product of two elements in that set whose norms are both greater than 1. Prove that every element in $\mathbb{Z}[\sqrt{5}]$ with absolute norm greater than 1 is equal to a product of irreducible elements. [*Note:* In this case one does NOT have a unique factorization result. In particular, we have $-4 = 2 \cdot (-2) = (1 + \sqrt{5})(1 - \sqrt{5})$ but neither $1 + \sqrt{5}$ nor $1 - \sqrt{5}$ can be written as $2u$ where $u \in \mathbb{Z}[\sqrt{5}]$ has absolute norm equal to 1.]

VI. Infinite constructions in set theory

VI.2. Infinite Cartesian products

101. Suppose that A is a set and $\{X_\alpha\}_{\alpha \in A}$ is an indexed family of sets with indexing set A. Prove that if X_β is empty for some $\beta \in A$, and X_γ is nonempty for some $\gamma \in A$, then the product $\prod_\alpha X_\alpha$ is also empty. [*Hint:* If x belongs to the product, what can we say about x_β?]

VI.3. Transfinite cardinal numbers

101. Strictly speaking, our definition of cardinal number depends upon choosing a large set containing everything else of interest. The purpose of this exercise is to show that the statement $|A| = |B|$ (*i.e.*, A and B have the same cardinality, does not depend upon the choice of the large universal set.

Let $\mathcal{U}$ be a large family of sets such that $A, B \in \mathcal{U}$, and let $\mathcal{W}$ be another family of sets with the same property. Explain why the statements

$$|A| = |B| \textit{ viewed as members of } \mathcal{U}$$

,

$$|A| = |B| \textit{ viewed as members of } \mathcal{W}$$

are logically equivalent.

102. Suppose that A, B, C are sets such that $|A| \leq |B| \leq |C|$ and $|A| = |C|$. Prove that $|A = |B| = |C|$.

103. Let A and B be sets such that $|A| < |B|$. Prove that there is a subset $A' \subset B$ such that $|A'| = |B|$.

VI.4. Countable and uncountable sets

101. Let A be a finite set (our "alphabet"). Then the set **String** (A) of *finite strings over* A is given by the union

$$\bigcup_{n=1}^{\infty} A^n \times \{n\}$$

where A^n denotes the n-fold product of A with itself and $\{n\}$ is appended to ensure that the copies of A^m and A^n are disjoint if $m \neq n$. Prove that **String** (A) is countably infinite.

VII. The Axiom of Choice and related topics

VII.1. Nonconstructive existence statements

101. Show that the set $\mathbb{R} - \mathbb{Q}$ of irrational numbers has the same cardinality as $\mathbb{R}$. [*Hint:* What is $\beta + \aleph_0$ if β is a transfinite cardinal?]

102. Given two positive integers $m < n$, let $G_m(\mathbb{R}^n)$ denote the set of vector subspaces $W \subset \mathbb{R}^n$ such that $\dim W = m$. Prove that $|G_m(\mathbb{R}^n)| = |\mathbb{R}^n|$.

103. Define an *algebraic hypersurface* in $\mathbb{R}^n$ to be the set of all points $(x_1, \cdots, x_n) \in \mathbb{R}^n$ such that $F(x_1, \cdots, x_n) = 0$ for some polynomial $F[t_1, \cdots, t_n]$ in n indeterminates (hence $F \in \mathbb{R}[t_1, \cdots, t_n]$). Prove that the cardinality of the set $\mathbf{H}$ of algebraic hypersurfaces in $\mathbb{R}^n$ is equal to $\mathbb{R}$.