Describing the inverse function to $f(x) = x + e^x$

We begin with a simple observation:

Example. *If $f(x) = x + e^x$, then f defines a 1–1 and onto function from the real line $\mathbb{R}$ to itself and therefore has an inverse function.*

Sketch of proof. The limit of $f(x)$ as $x \to \pm\infty$ is $\pm\infty$ (for $+\infty$ both summands go there, while for $-\infty$ the first summand does and the second goes to 0). Furthermore, $f(x)$ is strictly increasing because $f'(x) = 1 + e^x > 0$ for all x. Hence the Intermediate Value Theorem implies that f assumes all values between $-\infty$ and $+\infty$.■

As noted below, we cannot solve directly for the inverse function $g(y)$ such that $y = x + e^x$ using the standard functions from first year calculus, but the inverse can be described in terms of another much studied function known as *Lambert's W-function* (named after J. H. Lambert, who first discussed the function in the middle of the 18^{th} century). This function $w(z)$ is defined by the solution to the functional equation

$$w \exp(w) \quad = \quad z$$

and introductions to the basic properties of this function appear in

`http://en2.wikipedia.org/wiki/Lambert's_W_function`

and the paper by Corless *et al.* cited as a reference in that link. The following paper contains a proof that this function cannot be expressed in terms of the standard functions from first year calculus (another example of this sort appears in `nonelementary-integrals.pdf`):

> M. Bronstein, R. M. Corless, J. H. Davenport and D. J. Jeffrey, *Algebraic properties of the Lambert W-function from a result of Rosenlicht and of Liouville*, Integral Transforms and Special Functions **19** (2008), 709 – 712.

Although the Lambert W-function was first defined in the eighteenth century, there has been a great deal of renewed interest in it over the past two decades for several reasons: Advances in computer technology have made the function easier to analyze, the function is very useful in connection with computer software for symbolic manipulation of mathematical expressions, and there are several applications of this functions to other branches of science. Some additional links are listed below. The first of these includes a further link to numerous elementary exercises involving the Lambert W-function.

`http://www.apmaths.uwo.ca/~rcorless/frames/PAPERS/LambertW/`

`http://mathworld.wolfram.com/LambertW-Function.html`

`http://www.cecm.sfu.ca/publications/organic/rutgers/node34.htmlAC`

One can use Lambert's W-function to find a formula for the inverse to $x + e^x$ as follows: If we exponentiate the equation $x + e^x = y$ we obtain the equation

$$\exp(x) \cdot \exp\big(\exp(x)\big) \quad = \quad \exp(y)$$

and if we make the changes of variables $w = e^x$ and $z = e^y$ we obtain the identity $w\, e^w = z$ that defines the Lambert W-function. Solving the associated equation for x, we obtain the following formula for the inverse function:

$$g(y) \quad = \quad x \quad = \quad \log\big(W\,(e^y)\big)$$

This expresses x as a function of y very effectively, especially around the origin where one can write out a Taylor series expansion for $W(x)$ in a very neat explicit form:

$$W(x) \quad = \quad \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!}\, x^n$$