OUTLINE OF LOGIC LECTURES

Monday, October 2

Truth versus falsehood

Some statements to consider:

If n is an integer then n^2 is greater than or equal to n.
If n is an integer then n^3 is greater than or equal to n.
The square of an even integer is even.
The cube of an even integer is odd.

Statements and predicates

A declarative statement has two parts: A subject and a predicate. In a well-formed sentence, these must be unambiguous and meaningful with respect to each other.

Some non-well-formed examples:

The most visually appealing rectangle is one whose length L and width W satisfy the golden section equation L+W/L = L/W (the number phi, which is approximately 1.6180339887).

If we divide a circle by a line, the result is an irrational number.

Quantifiers

An upside down E means, “there exists,” or, “for some choice of.”
An upside down A means, “for all choices of.”

Compound statement using both:

For every positive real number a there is a positive real number b whose square is a.

NOTE that if we switch the two quantifiers in the preceding sentence, the resulting statement is false.

Logical implication

If P and Q are well-formed statements, then “P implies Q” means that either Q is true or P is false. This can lead to some bizarre implications that are “vacuously true.”

Validity of compound statements

There are many tautologies. For example, if P is true then (P or Q) is true, or if (P and Q) is true then P is true. One way to check the validity of a statement is to see what happens in every possibility. For example, if we have a statement involving substatements P, Q, R, S, then there are 16 possibilities to consider, depending upon whether P is true or false, and similarly for Q, R, S. These can be tabulated by sequences of 1's and 0's with four terms.

Wednesday, October 4

Negations of statements (implications, predicates, quantifiers):

When composing proofs, it is often crucial to know exactly what it means to say that a given statement is false.

Nonmathematical example of an incorrect negation:

If you don't support a total ban on Muslim immigration, you don't care if terrorists can enter the country freely.

For compound statements, the negation of "either P or Q is true" is "both P and Q are false," and the negation of "both P and Q are true" is "either P or Q is false."

For quantifiers, the negation of "for all relevant x, P(x) is true" is "for some relevant x, P(x) is false," and the negation of "for some relevant x, P(x) is true," is "for all relevant x, P(x) is false." In practice it is often necessary to look at compound quantifiers, as in "for each relevant x, there is a relevant y such that P(x,y) is true." What is the negation of this? "For some relevant x, there are no relevant choices of y such that P(x,y) is true."

Existence, uniqueness and counterexamples

To prove a statement like "There exists a unique relevant choice of x such that P(x) is true," it is necessary to find an example where the statement is true, and it is also necessary to show that if y and z are relevant choices such that P(y) and P(z) are true, then y and z must be the same; the two steps can be done in either order.

Example: To prove that there is a unique real number Z such that xZ=Z for all x, one step is to note this is true if Z = 0, and the other is to prove that if Z does have this property then Z=0.

Frequently the most direct way of proving a statement P(x) is not always true is to describe an example b such that P(b) is false. For example, consider the statement, "If x is a positive real number then x^2 is greater than or equal to x." In order to prove this is false, it is only necessary to find some positive value of x such that x^2 is strictly less than x. One obvious choice is x = ½.

Direct proofs, contrapositives and *reductio ad absurdum*

The simplest arguments are those which are a sequence of direct logical steps starting from the hypothesis and ending with the conclusion. For example, consider the algebraic result, "If 2x+3=5 then x=1." To prove this we subtract 3 from both sides, obtaining 2x=2, and then we divide both sides by 2 and conclude that x=1.

Sometimes it is better to rephrase things, and one way of doing so is by taking CONTRAPOSITIVES. This means we replace the statement, "If P is true, then Q is true," with the logically equivalent statement, "If Q is false, then P is false." Logical equivalence means that either both statements are true or both statements are false.

Example: Consider the statement, "If L and M are two distinct lines in space, then they have at most one point in common." The contrapositive statement is that if L and M are lines in space with at least two points in common, then L and M must be equal. In this case the contrpositive is easier to analyze, for we know that if we are given two points there is a unique line between them.

WARNING: Usually a statement of the form, "If P is true then Q is true," is NOT logically equivalent to either its CONVERSE, "If Q is true then P is true," or its INVERSE, "If P is false then Q is false." However, each of the latter statements is the contrapositive of the other, so either both are true or both are false.

Another indirect method of proof is by contradiction or *reductio ad absurdum*. The objective is again to prove the statement, "If P is true then Q is true," and the idea is to assume that the conclusion Q is false, and to proceed by using this extra tool to derive some sort of contradiction. Proof by taking constrapositives can be viewed as a particular case of this in which the contradiction is that P is both true (by assumption) and false (by the reasoning based on the assumption that Q is false). However, there may be other ways in which a contradiction can be derived.

Example: Consider the standard and ancient proof that the square root of 2 is not a rational number. Suppose it indeed a rational number and write this number in the form p/q where p and q are positive integers with no common positive factors other than 1. Subsequent steps in the argument show that first p and then q must be even. This is a contradiction because we choose p and q so that they did not have any factors in common except for the obvious choice of 1.

Further examples and more detailed information on these topics also appear in the course directory files mathproofs.pdf, mathproofs2.pdf, polya.pdf, and solutions90.pdf.