AMBIGUITY PROBLEMS WITH POLAR COORDINATES

A given point in the plane can be represented by more than one pair of polar coordinates, and Section 9.7 of Thomas and Finney, *Calculus, Ninth Edition*, discusses two problems related to this fact.

(1) If we are given a curve C defined by an equation $F(r,\theta) = 0$ and a point P with polar coordinates (q,α), then P may lie on C even if $F(q,\alpha) \neq 0$.

(2) If we are given two curves A and B defined by equations $F(r,\theta) = 0$ and $G(r,\theta) = 0$, then there may be hidden points (q,α) lying on BOTH curves A and B such that (q,α) is NOT a simultaneous solution of the given two equations. In other words, at least one of $F(q,\alpha)$ or $G(q,\alpha)$ is nonzero.

Specific examples are given on pages 760 – 762 in the Ninth Edition of Thomas and Finney, which addresses the second issue by suggesting that one graph the curves A and B to see if there are any common points that are not given by simultaneous solutions of the equations. This is usually effective, but it is not systematic or logically complete. We shall describe an analytic procedure for finding all such hidden points and use it to solve examples like those in the text.

Copies of the relevant pages from Thomas and Finney are appended at the endof this file.

Points with the same polar coordinates

Since the problems in (1) and (2) arise because a point is representable by more than one set of polar coordinates, it is best to begin by recalling when two pairs of polar coordinates (q,α) and (s,β) define the same point in the plane. There are two distinct criteria:

(i) Both q and s are nonzero, and there is an integer n such that $\beta = \alpha + n\,\pi$ and $s = (-1)^n q$.

(ii) Both q and s are zero, and α and β are arbitrary.

These criteria are exactly the conditions under which the‘ equations $q\cos\alpha = r\cos\alpha$ and $q\sin\alpha = r\sin\alpha$ are both satisfied.■

Application to ambiguity issues

Here is the general criterion for determining whether a point with polar coordinates (q,α) lies on the curve C defined by $F(r,\theta) = 0$.

General condition for (1)**:** *If the curve C is defined by $F(r,\theta) = 0$, then the point P with polar coordinates (q,α) lies on C if and only if one of the following is true:* (a) *We have $q \neq 0$ and there is an n such that $F(\,(-1)^n q, \alpha + n\pi) = 0$.* ($b$) *We have $q = 0$ and there is some β such that $F(q,\beta) = 0$.*■

General condition for (2)**:** *If the curves sA and B are defined by $F(r,\theta) = 0$ and $G(r,\theta) = 0$ respectively, then the point P with polar coordinates (q,α) lies on both A and B if and only if one of the following is true:* (a) *We have $q \neq 0$ and there are integers m and n such that $F(\,(-1)^m q, \alpha + m\pi) = 0$ and $G(\,(-1)^n q, \alpha + n\pi) = 0$.* ($b$) *We have $q = 0$ and there are some β and γ such that $F(q,\beta) = 0$ and $F(q,\gamma) = 0$.*■

Textbook examples and exercises usually have a crucial property which leads to simplified criteria. In such examples the function $H = F$ and (if applicable) G satisfy the **periodicity property**

$$H(r,\theta) \quad = \quad H(r,\theta + 2\pi)$$

for all r and θ. In particular, this holds when H can be written as a sum of terms of the form $f(r)\sin^a\theta\cos^b\theta$ where a and b run through finite sets of integers (note that functions like $\sin k\theta$ and $\cos \ell\theta$ are all polynomials in $\sin\theta$ and $\cos\theta$). In such cases there are only two possibilities for the values of the expressions $H(\,(-1)^n q, \theta + n\pi)$, and one can simplify the criterion (a) in the general conditions as follows:

Specialized condition for (1)**:** *If the curve C is defined by $F(r,\theta) = 0$ where F has the periodicity property, then the point P with polar coordinates (q,α) lies on C if and only if one of the following is true: (a) We have $q \neq 0$ and either $F(q,\alpha) = 0$ or else $F(-q, \alpha+\pi) = 0$. (b) We have $q = 0$ and there is some β such that $F(q,\beta) = 0$.*■

Specialized condition for (2)**:** *If the curves A and B are defined by $F(r,\theta) = 0$ and $G(r,\theta) = 0$ respectively, then the point P with polar coordinates (q,α) lies on both A and B if and only if one of the following is true: (a) We have $q \neq 0$ and either $F(q,\alpha) = G(q,\alpha) = 0$ or $F(q,\alpha) = G(-q,\alpha+\pi) = 0$ or $F(-q,\alpha) = G(q,\alpha) = 0$. (b) We have $q = 0$ and there are some β and γ such that $F(q,\beta) = 0$ and $F(q,\gamma) = 0$.*■

Proof of the main result(s)

Let $\mathbf{Rect} : \mathbb{R}^2 \to \mathbb{R}^2$ be the usual map from polar to rectangular coordinates which sends (r,θ) to $(r\cos\theta, r\sin\theta)$. To prove (1), note that $\mathbf{Rect}(s,\alpha)$ lies on the curve defined by $F(r,\theta) = 0$ if and only if there is some (t,β) such that $\mathbf{Rect}(s,\alpha) = \mathbf{Rect}(t,\beta)$ and $F(t,\beta) = 0$. The conditions for $\mathbf{Rect}(s,\alpha) = \mathbf{Rect}(t,\beta)$ are either $s = t = 0$ or $s,t \neq 0$ and there is some integer n such that $t = (-1)^n s$ and $\beta = \alpha + n\pi$). The conclusion of (1) follows by substituting the conditions of the second sentence into the first.

To prove (2), recall that the condition for a point $\mathbf{Rect}(s,\alpha)$ to be on both curves is that there are ordered pairs (s_1,α_1) and (s_2,α_2) which represent the same point in rectangular coordinates and satisfy the equations $F(s_1,\alpha_1) = 0$ and $G(s_2,\alpha_2) = 0$. There are now two cases.

Suppose that $\mathbf{Rect}(s,\alpha) = (0,0)$. Then $s = s_1 = s_2 = 0$ and the condition for the origin to be a point on both curves is that there are angles β and γ such that $F(0,\beta) = 0$ and $G(0,\gamma) = 0$.

Now suppose that $\mathbf{Rect}(s,\alpha) \neq (0,0)$. Then the condition above translates into the condition that there are integers m and n such that $F(\,(-1)^m s, \alpha + m\pi) = 0$ and $G(\,(-1)^n s, \alpha + n\pi) = 0$.■

The specialized statements now follow from the previous discussion.■

Application to ambiguity issues

We shall now apply the preceding criteria to the examples worked out on pages 760 – 762 of the book by Thomas and Finney, and also to solve exercises 67 – 68 which are described on page 789 of Scharf and Weir, *Instructor's Solutions Manual Part* I *to accompany Thomas' Calculus, Early Transcendentals, Tenth Edition*. A copy of that page is also appended at the end of this file.

EXAMPLE 5. We are given the curve $F(r,\theta) = r - 2\cos 2\theta = 0$ and we want to show that the point with polar coordinates $(2, \pi/2)$ lies on it. According to the first specialized condition, this will hold if either $F(2,\pi/2) = 0$ or $F(-2, 3\pi/2) = 0$. The first of these does not hold, but the second does because $F(-2, 3\pi/2)$ is equal to $F(-2, -\pi/2)$, and as noted in the text the latter turns out to be zero.■

EXAMPLE 6. We want to find the common points to the curves defined by $F(r,\theta) = r^2 - 4\cos\theta = 0$ and $G(r,\theta) = r - 1 + \cos\theta = 0$. According to the second specialized condition, there are four parts to this, the first three of which are solving the three pairs of simultaneous equations as in (a) and the last of which involves cases where $r = 0$.

One begins with the system of equations $F(q, \alpha) = G(q, \alpha) = 0$ as in the text, and this yields two common points. Next, one considers the system $F(q, \alpha) = G(-q, \alpha + \pi) = 0$, which reduces to $r^2 = 4\cos\theta$ and $-r = 1 + \cos\theta$. Following the same pattern as on page 767 of the text, we obtain the equation

$$-r \quad = \quad 1 \; + \; \cos\theta \quad = \quad 1 \; + \; \frac{r^2}{4}$$

which further reduces to $0 = r^2 + 4r + 1$, so that $r = \pm 2$. For these choices of r we have

$$\mp 2 \quad = \quad 1 \; + \; \cos\theta$$

so that $-1 \mp 2 = \cos\theta$. Since $-3 = \cos\theta$ is impossible, we are left with the case where $r = -2$ and $1 = \cos\theta$, so that the point with polar coordinates $(-2, 0)$ also lies on the curve. Since the polar coordinates $(-2, 0)$ and $(2, \pi)$ determine the same point, we conclude that $(2, \pi)$ is a hidden intersection point of A and B, as noted in the last paragraph of page 767.

The third part of the problem in our procedure is to examine the system $F(-q, \alpha + \pi) = G(q, \alpha) = 0$, and in this case the first equation is equivalent to $r^2 = -4\cos\theta$. If we square the second equation we obtain $r^2 = (1 - \cos\theta)^2$, and if we combine the preceding two equations we find that

$$-4\cos\theta \quad - \quad 1 \; - \; 2\cos\theta \; + \; \cos^2\theta$$

which is equivalent to $0 = (1 + \cos\theta)^2$. This implies that $\theta = \pi$ and $r = 0$. Strictly speaking, this means that we have no solutions corresponding to the third part of the problem because at this point we are only looking for solutions where the first polar coordinate is nonzero.

Finally, the fourth step is the one which decides whether or not the origin or pole lies on both curves. All we need to do is find α and β such that $F(0, \alpha) = 0$ and $G(0, \beta) = 0$. But we see directly that $F(0, \pi/2) = 0$ and $G(0, 0) = 0$, and therefore it follows that $(0, 0)$ is another hidden intersection point of A and B. Furthermore, this also shows that there are no other common points aside from the four that were found in the text.■

EXERCISE 67. Here we want to find the intersection points of the two cardioids $r = 1 \pm \cos\theta$. Graphing these curves suggests that there are three common points, one at the origin and two others at points on the y-axis that are symmetric with respect to the origin. We need to show that our procedure yields all these points and no others.

The first step is to solve the simultaneous equations $0 = F(r, \theta) = r - 1 + \cos\theta$ and $0 = G(r, \theta) = r - 1 - \cos\theta$ when $r \neq 0$. These yield $\cos\theta = 0$ and hence that $\theta = \pi/2$ or $3\pi/2$ and consequently $r = 1$. Thus we have checked that the two points with polar coordinates $(1, \pm\pi/2)$ lie on the curve.

Next we consider the system $0 = F(r, \theta) = G(-r, \theta + \pi)$, The second equation is just $0 = -r - 1 + \cos\theta$, so we have the system of equations $r = 1 - \cos\theta = -1 + \cos\theta$. These yield the impossible equation $\cos\theta = 2$, so there are no hidden points given by simultaneous solutions of the second system.

Turing to the third system $0 = G(r, \theta) = F(-r, \theta + \pi)$, we have $F(-r, \theta + \pi) = -r - 1 - \cos\theta$, so this system becomes $r = -1 - \cos\theta = 1 + \cos\theta$. These yield the impossible equation $\cos\theta = -2$, so there are no hidden points given by simultaneous solutions of the third system.

Finally, we must check to see whether there exist α and β such that $F(0,\alpha) = G(0,\beta) = 0$. The answer is yes, and we can choose α and β to be 0 and π respectively.■

EXERCISE 68. Here we have the circle $r = 2\sin\theta$ whose radius is 1 and whose center has rectangular coordinates $(0,1)$, and we also have the four leaf curve $r = 2\sin 2\theta$ whose center is the origin and whose four leaves each lie in different quadrants of the coordinate plane.

We want to find the common points of the curves whose equations are $F(r,\theta) = r - 2\sin\theta$ and $G(r,\theta) = r - 2\sin 2\theta$. These yield the equation $\sin\theta = \sin 2\theta$, and if we solve these for θ we find that $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$. For the first option we have $\theta = 0$ or π, and for the second we have $\theta = \pm\pi/3$. The first option $\theta = 0$ or π implies $r = 0$, and although this is not strictly covered by the conditions in the first part of the procedure we can see that $F(0,0) = G(0,0) = 0$, so that $(0,0)$ lies on both curves. This means we can skip the fourth part of the procedure. Turning to the second option, it implies that $r = \pm\sqrt{3}$, and therefore we also see that the points with polar coordinates $(\pm\sqrt{3}, \pm\pi/3)$ also lie on both curves.

The second part of the problem involves the equations $F(q,\alpha) = G(-q,\alpha+\pi) = 0$, which leads to the equation $\sin\theta = -\sin 2\theta$. The solutions to this equation are $\theta = 0$ or π and $\theta = \pm 2\pi/3$. We have already considered the first option, and the second option again leads to $r = \pm\sqrt{3}$, sop that we obtain the points with polar coordinates $(\pm\sqrt{3}, \pm 2\pi/3)$ as common points of the curve. However, these are not new because one can check that $(\pm\sqrt{3}, \pm 2\pi/3)$ represent the same points as $(\mp\sqrt{3}, \mp\pi/3)$ respectively.

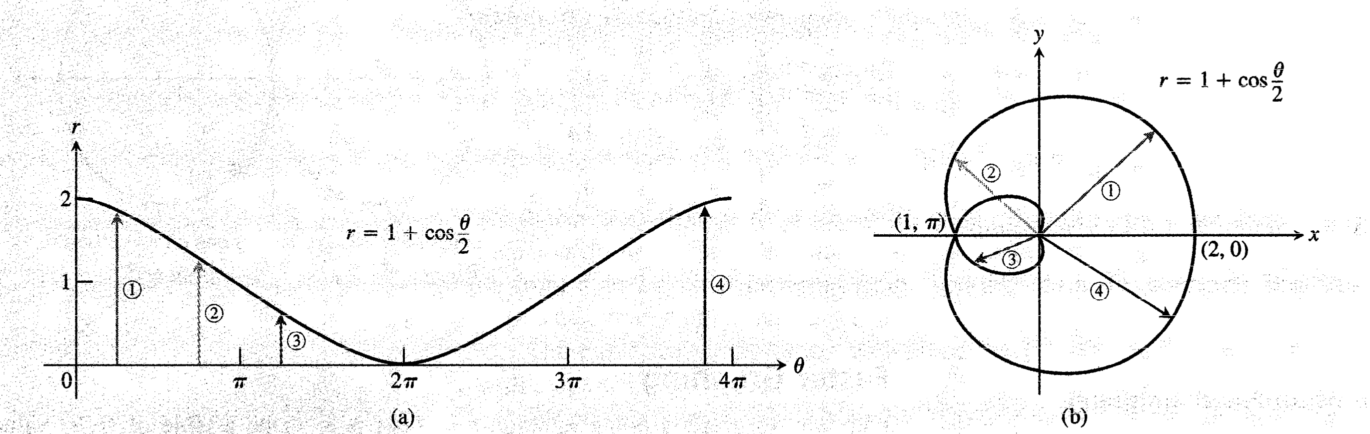
Finally, the third part of the problem involves the equations $F(-q,\alpha) = G(q,\alpha) = 0$, which leads to the equation $-\sin\theta = \sin 2\theta$. This is equivalent to the equation considered in the second part of the problem, so it will yield nothing new. Therefore all the solutions are given by the (extended) first part of the problem as above.■

AMBIGUITY PROBLEMS WITH POLAR COORDINATES

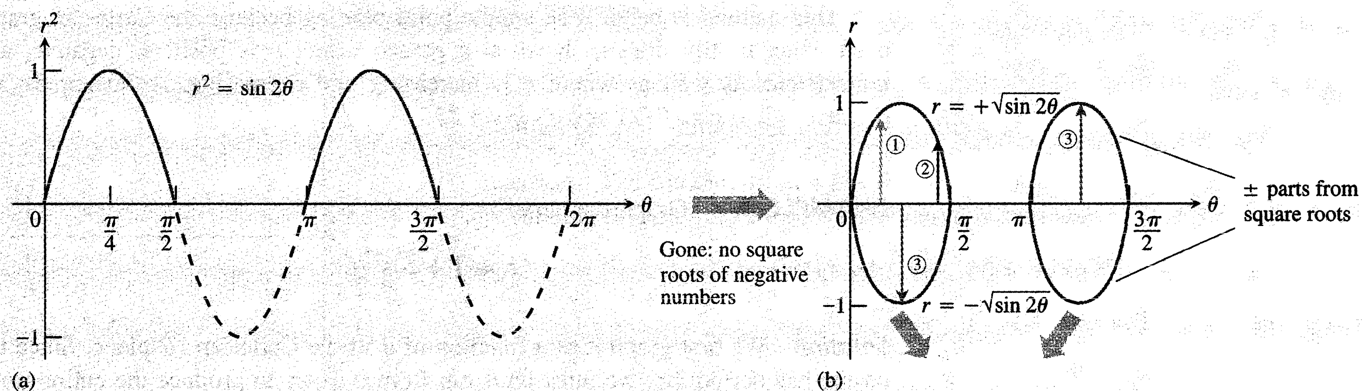
The relevant pages from the two cited references are attached.

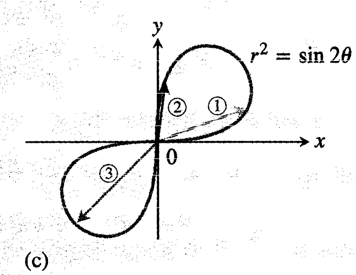
Drawing Lesson

How to Use Cartesian Graphs to Draw Polar Graphs



9.56 (a) The graph of $r = 1 + \cos(\theta/2)$ in the Cartesian $r\theta$-plane gives the radii for the graph in the polar $r\theta$-plane (b).





9.57 (a) The graph of $r^2 = \sin 2\theta$ in the Cartesian $r^2\theta$-plane includes negative values of the dependent variable r^2 as well as positive values. (b) When we graph r vs. θ in the Cartesian $r\theta$-plane, we ignore the points where r is imaginary but plot + and − parts from the points where r^2 is positive. (c) In the polar $r\theta$-plane, the radii from the previous sketch cover the final graph twice.

Finding Points Where Polar Graphs Intersect

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve.

Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. The only sure way to identify all the points of intersection is to graph the equations.

EXAMPLE 5 *Deceptive coordinates*

Show that the point $(2, \pi/2)$ lies on the curve $r = 2 \cos 2\theta$.

Solution It may seem at first that the point $(2, \pi/2)$ does not lie on the curve because substituting the given coordinates into the equation gives

$$2 = 2 \cos 2\left(\frac{\pi}{2}\right) = 2 \cos \pi = -2,$$

which is not a true equality. The magnitude is right, but the sign is wrong. This suggests looking for a pair of coordinates for the given point in which r is negative, for example, $(-2, -(\pi/2))$. If we try these in the equation $r = 2 \cos 2\theta$, we find

$$-2 = 2 \cos 2\left(-\frac{\pi}{2}\right) = 2(-1) = -2,$$

and the equation is satisfied. The point $(2, \pi/2)$ does lie on the curve. ❑

EXAMPLE 6 *Elusive intersection points*

Find the points of intersection of the curves

$$r^2 = 4 \cos \theta \qquad \text{and} \qquad r = 1 - \cos \theta.$$

Solution In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others are found by graphing. (Also, see Exercise 49.)

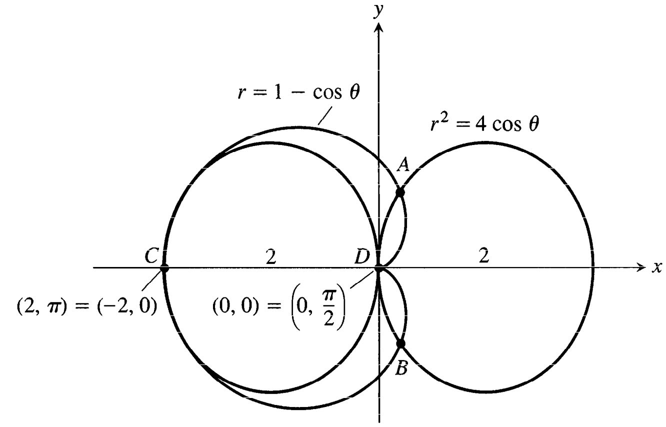
If we substitute $\cos \theta = r^2/4$ in the equation $r = 1 - \cos \theta$, we get

$$\begin{aligned} r &= 1 - \cos \theta = 1 - \frac{r^2}{4} \\ 4r &= 4 - r^2 \\ r^2 + 4r - 4 &= 0 \\ r &= -2 \pm 2\sqrt{2}. && \text{Quadratic formula} \end{aligned}$$

The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to belong to either curve. The values of θ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\begin{aligned} \theta &= \cos^{-1}(1 - r) && \text{From } r = 1 - \cos \theta \\ &= \cos^{-1}\left(1 - \left(2\sqrt{2} - 2\right)\right) && \text{Set } r = 2\sqrt{2} - 2. \\ &= \cos^{-1}\left(3 - 2\sqrt{2}\right) \\ &= \pm 80^\circ. && \text{Rounded to the nearest degree} \end{aligned}$$

We have thus identified two intersection points: $(r, \theta) = (2\sqrt{2} - 2, \pm 80^\circ)$.



9.58 The four points of intersection of the curves $r = 1 - \cos\theta$ and $r^2 = 4\cos\theta$ (Example 6). Only A and B were found by simultaneous solution. The other two were disclosed by graphing.

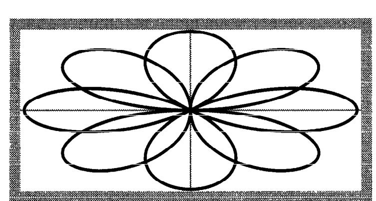
If we graph the equations $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$ together (Fig. 9.58), as we can now do by combining the graphs in Figs. 9.54 and 9.55, we see that the curves also intersect at the point $(2, \pi)$ and the origin. Why weren't the r-values of these points revealed by the simultaneous solution? The answer is that the points $(0, 0)$ and $(2, \pi)$ are not on the curves "simultaneously." They are not reached at the same value of θ. On the curve $r = 1 - \cos\theta$, the point $(2, \pi)$ is reached when $\theta = \pi$. On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$, where it is identified not by the coordinates $(2, \pi)$, which do not satisfy the equation, but by the coordinates $(-2, 0)$, which do. Similarly, the cardioid reaches the origin when $\theta = 0$, but the curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$. ❑

Technology *Finding Intersections* The *simultaneous mode* of a graphing utility gives new meaning to the *simultaneous solution* of a pair of polar coordinate equations. A simultaneous solution occurs only where the two graphs "collide" while they are being drawn simultaneously and not where one graph intersects the other at a point that had been illuminated earlier. The distinction is particularly important in the areas of traffic control or missile defense. For example, in traffic control the only issue is whether two aircraft are in the same place at the same time. The question of whether the curves the craft follow intersect is unimportant.

To illustrate, graph the polar equations

$$r = \cos 2\theta \qquad \text{and} \qquad r = \sin 2\theta$$

in simultaneous mode with $0 \le \theta < 2\pi$, θ Step $= 0.1$, and view dimensions [xmin, xmax] $= [-1, 1]$ by [ymin, ymax] $= [-1, 1]$. *While the graphs are being drawn on the screen,* count the number of times the two graphs illuminate a single pixel simultaneously. Explain why these points of intersection of the two graphs correspond to simultaneous solutions of the equations. (You may find it helpful to slow down the graphing by making θ Step smaller, say 0.05, for example.) In how many points total do the graphs actually intersect?



$r_1 = \sin 2\theta$ and $r_2 = \cos 2\theta$ graphed together.

Exercises 9.7

Symmetries and Polar Graphs

Identify the symmetries of the curves in Exercises 1–12. Then sketch the curves.

1. $r = 1 + \cos\theta$

2. $r = 2 - 2\cos\theta$

3. $r = 1 - \sin\theta$

4. $r = 1 + \sin\theta$

5. $r = 2 + \sin\theta$

6. $r = 1 + 2\sin\theta$

7. $r = \sin(\theta/2)$

8. $r = \cos(\theta/2)$

9. $r^2 = \cos\theta$

10. $r^2 = \sin\theta$

11. $r^2 = -\sin\theta$

12. $r^2 = -\cos\theta$

Graph the lemniscates in Exercises 13–16. What symmetries do these curves have?

13. $r^2 = 4\cos 2\theta$

14. $r^2 = 4\sin 2\theta$

15. $r^2 = -\sin 2\theta$

16. $r^2 = -\cos 2\theta$

Slopes of Polar Curves

Use Eq. (1) to find the slopes of the curves in Exercises 17–20 at the given points. Sketch the curves along with their tangents at these points.

17. *Cardioid.* $r = -1 + \cos\theta$; $\theta = \pm\pi/2$

18. *Cardioid.* $r = -1 + \sin\theta$; $\theta = 0, \pi$

19. *Four-leaved rose.* $r = \sin 2\theta$; $\theta = \pm\pi/4, \pm 3\pi/4$

20. *Four-leaved rose.* $r = \cos 2\theta$; $\theta = 0, \pm\pi/2, \pi$

Limaçons

Graph the limaçons in Exercises 21–24. Limaçon ("*lee*-ma-sahn") is Old French for "snail." You will understand the name when you graph the limaçons in Exercise 21. Equations for limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. There are four basic shapes.

21. *Limaçons with an inner loop*

a) $r = \dfrac{1}{2} + \cos\theta$ **b)** $r = \dfrac{1}{2} + \sin\theta$

22. *Cardioids*

a) $r = 1 - \cos\theta$ **b)** $r = -1 + \sin\theta$

23. *Dimpled limaçons*

a) $r = \dfrac{3}{2} + \cos\theta$ **b)** $r = \dfrac{3}{2} - \sin\theta$

24. *Oval limaçons*

a) $r = 2 + \cos\theta$ **b)** $r = -2 + \sin\theta$

Graphing Polar Inequalities

25. Sketch the region defined by the inequalities $-1 \le r \le 2$ and $-\pi/2 \le \theta \le \pi/2$.

26. Sketch the region defined by the inequalities $0 \le r \le 2\sec\theta$ and $-\pi/4 \le \theta \le \pi/4$.

In Exercises 27 and 28, sketch the region defined by the inequality.

27. $0 \le r \le 2 - 2\cos\theta$

28. $0 \le r^2 \le \cos\theta$

Intersections

29. Show that the point $(2, 3\pi/4)$ lies on the curve $r = 2\sin 2\theta$.

30. Show that $(1/2, 3\pi/2)$ lies on the curve $r = -\sin(\theta/3)$.

Find the points of intersection of the pairs of curves in Exercises 31–38.

31. $r = 1 + \cos\theta,\quad r = 1 - \cos\theta$

32. $r = 1 + \sin\theta,\quad r = 1 - \sin\theta$

33. $r = 2\sin\theta,\quad r = 2\sin 2\theta$

34. $r = \cos\theta,\quad r = 1 - \cos\theta$

35. $r = \sqrt{2},\quad r^2 = 4\sin\theta$

36. $r^2 = \sqrt{2}\sin\theta,\quad r^2 = \sqrt{2}\cos\theta$

37. $r = 1,\quad r^2 = 2\sin 2\theta$

38. $r^2 = \sqrt{2}\cos 2\theta,\quad r^2 = \sqrt{2}\sin 2\theta$

GRAPHER Find the points of intersection of the pairs of curves in Exercises 39–42.

39. $r^2 = \sin 2\theta,\quad r^2 = \cos 2\theta$

40. $r = 1 + \cos\dfrac{\theta}{2},\quad r = 1 - \sin\dfrac{\theta}{2}$

41. $r = 1,\quad r = 2\sin 2\theta$

42. $r = 1,\quad r^2 = 2\sin 2\theta$

Grapher Explorations

43. Which of the following has the same graph as $r = 1 - \cos\theta$?

a) $r = -1 - \cos\theta$
b) $r = 1 + \cos\theta$

Confirm your answer with algebra.

44. Which of the following has the same graph as $r = \cos 2\theta$?

a) $r = -\sin(2\theta + \pi/2)$
b) $r = -\cos(\theta/2)$

Confirm your answer with algebra.

45. *A rose within a rose.* Graph the equation $r = 1 - 2\sin 3\theta$.

46. *The nephroid of Freeth.* Graph the nephroid of Freeth:

$$r = 1 + 2\sin\frac{\theta}{2}.$$

47. *Roses.* Graph the roses $r = \cos m\theta$ for $m = 1/3$, 2, 3, and 7.

48. *Spirals.* Polar coordinates are just the thing for defining spirals. Graph the following spirals.

a) $r = \theta$

b) $r = -\theta$

c) *A logarithmic spiral:* $r = e^{\theta/10}$

d) *A hyperbolic spiral:* $r = 8/\theta$

e) *An equilateral hyperbola:* $r = \pm 10/\sqrt{\theta}$

(Use different colors for the two branches.)

Theory and Examples

49. (*Continuation of Example 6.*) The simultaneous solution of the equations

$$r^2 = 4\cos\theta \tag{2}$$

$$r = 1 - \cos\theta \tag{3}$$

in the text did not reveal the points (0, 0) and $(2, \pi)$ in which their graphs intersected.

a) We could have found the point $(2, \pi)$, however, by replacing the (r, θ) in Eq. (2) by the equivalent $(-r, \theta + \pi)$ to obtain

$$\begin{aligned} r^2 &= 4\cos\theta \\ (-r)^2 &= 4\cos(\theta + \pi) \\ r^2 &= -4\cos\theta. \end{aligned} \tag{4}$$

Solve Eqs. (3) and (4) simultaneously to show that $(2, \pi)$ is a common solution. (This will still not reveal that the graphs intersect at (0, 0).)

b) The origin is still a special case. (It often is.) Here is one way to handle it: Set $r = 0$ in Eqs. (2) and (3) and solve each equation for a corresponding value of θ. Since $(0, \theta)$ is the origin for *any* θ, this will show that both curves pass through the origin even if they do so for different θ-values.

50. If a curve has any two of the symmetries listed at the beginning of the section, can anything be said about its having or not having the third symmetry? Give reasons for your answer.

*51. Find the maximum width of the petal of the four-leaved rose $r = \cos 2\theta$, which lies along the x-axis.

*52. Find the maximum height above the x-axis of the cardioid $r = 2(1 + \cos\theta)$.

9.8 Polar Equations for Conic Sections

Polar coordinates are important in astronomy and astronautical engineering because the ellipses, parabolas, and hyperbolas along which satellites, moons, planets, and comets move can all be described with a single relatively simple coordinate equation. We develop that equation here.

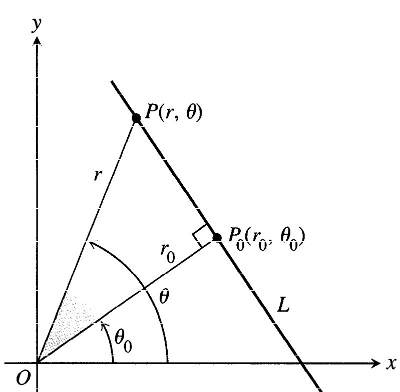
Lines

Suppose the perpendicular from the origin to line L meets L at the point $P_0(r_0, \theta_0)$, with $r_0 \geq 0$ (Fig. 9.59). Then, if $P(r, \theta)$ is any other point on L, the points P, P_0, and O are the vertices of a right triangle, from which we can read the relation

$$\frac{r_0}{r} = \cos(\theta - \theta_0)$$

or

$$r\cos(\theta - \theta_0) = r_0.$$



9.59 We can obtain a polar equation for line L by reading the relation $r_0/r = \cos(\theta - \theta_0)$ from triangle OP_0P.

The Standard Polar Equation for Lines

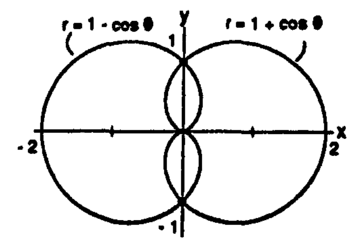
If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then an equation for L is

$$r\cos(\theta - \theta_0) = r_0.$$

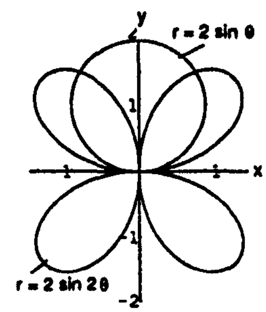
65. $\left(2, \frac{3\pi}{4}\right)$ is the same point as $\left(-2, -\frac{\pi}{4}\right)$; $r = 2 \sin 2\left(-\frac{\pi}{4}\right) = 2 \sin\left(-\frac{\pi}{2}\right) = -2 \Rightarrow \left(-2, -\frac{\pi}{4}\right)$ is on the graph $\Rightarrow \left(2, \frac{3\pi}{4}\right)$ is on the graph

66. $\left(\frac{1}{2}, \frac{3\pi}{2}\right)$ is the same point as $\left(-\frac{1}{2}, \frac{\pi}{2}\right)$; $r = -\sin\left(\frac{\left(\frac{\pi}{2}\right)}{3}\right) = -\sin \frac{\pi}{6} = -\frac{1}{2} \Rightarrow \left(-\frac{1}{2}, \frac{\pi}{2}\right)$ is on the graph $\Rightarrow \left(\frac{1}{2}, \frac{3\pi}{2}\right)$ is on the graph

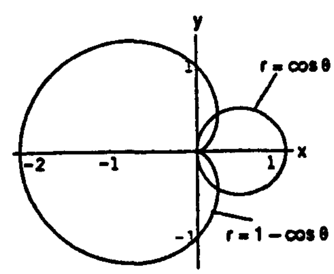
67. $1 + \cos\theta = 1 - \cos\theta \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow r = 1$; points of intersection are $\left(1, \frac{\pi}{2}\right)$ and $\left(1, \frac{3\pi}{2}\right)$. The point of intersection $(0,0)$ is found by graphing.



68. $2 \sin\theta = 2 \sin 2\theta \Rightarrow \sin\theta = \sin 2\theta \Rightarrow \sin\theta = 2 \sin\theta \cos\theta \Rightarrow \sin\theta - 2 \sin\theta \cos\theta = 0 \Rightarrow (\sin\theta)(1 - 2\cos\theta) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2} \Rightarrow \theta = 0, \frac{\pi}{3}$, or $-\frac{\pi}{3}$; $\theta = 0 \Rightarrow r = 0$, $\theta = \frac{\pi}{3} \Rightarrow r = \sqrt{3}$, and $\theta = -\frac{\pi}{3} \Rightarrow r = -\sqrt{3}$; points of intersection are $(0,0)$, $\left(\sqrt{3}, \frac{\pi}{3}\right)$, and $\left(-\sqrt{3}, -\frac{\pi}{3}\right)$



69. $\cos\theta = 1 - \cos\theta \Rightarrow 2\cos\theta = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, -\frac{\pi}{3} \Rightarrow r = \frac{1}{2}$; points of intersection are $\left(\frac{1}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{1}{2}, -\frac{\pi}{3}\right)$. The point $(0,0)$ is found by graphing.



70. $1 = 2 \sin 2\theta \Rightarrow \sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$; points of intersection are $\left(1, \frac{\pi}{12}\right)$, $\left(1, \frac{5\pi}{12}\right)$, $\left(1, \frac{13\pi}{12}\right)$, and $\left(1, \frac{17\pi}{12}\right)$. No other points are found by graphing.