Supplement to IV.2 — Product orderings

This is a written version of a result that was covered in class but not in the lecture notes.

Definition. Let A and B be partially ordered sets. Define a binary relation P on $A \times B$ by $(a,b)P(a',b')$ if and only if $a \leq a'$ and $b \leq b'$.

The relstion P is called the *product partial ordering* on $A \times B$, and this usage is justified by the following result:

THEOREM. *If A and B are partially ordered sets as above, then P is a partial ordering on $A \times B$.*

Proof. The reflexive property $(a,b)P(a,b)$ follows from $a \leq a$ and $b \leq b$. To see that P is symmetric, suppose that $(a,b)P(a',b')$ and $(a',b')P(a,b)$. Then we have $a \leq a'$ and $a' \leq a$, so that $a = a'$. Similarly, we have $b \leq b'$ and $b' \leq b$, so that $b = b'$; combining these, we conclude that $(a,b) = (a',b')$. Finally, to show P is transitive, suppose that $(a,b)P(a',b')$ and $(a',b')P(a'',b'')$. Then $a \leq a'$ and $a' \leq a''$ imply $a \leq a''$. Similarly, $b \leq b'$ and $b' \leq b''$ imply $b \leq b''$. Combining these, we have $(a,b)P(a'',b'') = (a'',b'')$. This completes the proof that P defines a partial ordering on $A \times B$.■

It is natural to ask how the product ordering P is related to the lexicographic orderintg L constructed in the notes. Here is a partial answer.

THEOREM. *If L and P respectively denote the lexicographic and partial orderings on $A \times B$, then $(a,b)P(a',b')$ implies $(a,b)L(a',b')$.*

In a situation like this one often says that the partial ordering L is a **refinement** of the partial ordering P.

Proof. If $(a,b)P(a',b')$, then $a \leq a'$. If $a < a'$, then by definition we have $(a,b)L(a',b')$. On the other hand if $a = a'$, then since $b \leq b'$ we have $(a,b)L(a,b') = (a',b')$.■

Example. Usually the lexicographic ordering is a **strict** refinement of the product ordering; *i.e.*, there are pairs (a,b) and (c,d) such that $(a,b)L(c,d)$ is true but $(a,b)P(c,d)$ is false. Consider the linearly ordered set consisting of the ordinary alphabet

$$A \quad = \quad \{a,b,c,d,e,f,g,h,i,j,k,l,m,n,o,p,q,r,s,t,u,v,w,x,y,z\}$$

with the usual alphabetical ordering. Then (a,z) precedes (b,a) in the lexicographic ordering but not in the product ordering. Of course, (b,a) does not precede (a,z) in either ordering.