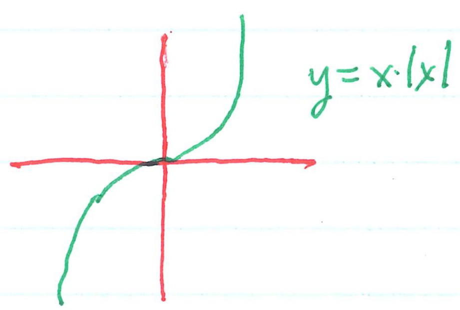
Quiz 2 Preparation

1. Each statement is the contrapositive of the other, so if one is true then so is the other. $\blacksquare$

2. (a) f is 1-1 since $2p = 2q \Rightarrow p = q$, but f is not onto; for example, $1 \notin f[\mathbb{Z}]$.
(b) Since $f[\mathbb{Z}] = 2\mathbb{Z}$, if $g : \mathbb{Z} \to 2\mathbb{Z}$ sends n to $2n$, then g is onto. It is also 1-1 because f is 1-1. $\blacksquare$

3. One way to analyze this problem is to start by graphing the function



so $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x \leq 0 \end{cases}$

The description on the right shows that $f'(x) = 2|x|$, so $f'(x) > 0$ if $x \neq 0$. In any case, we see that f is a strictly increasing continuous function from $(-\infty, 0]$ to itself and from $[0, \infty)$ to itself. Hence f is both 1-1 and onto. $\blacksquare$

4. The argument only shows that there is a unique real number $4x+3$ associated to each $x \in \mathbb{R}$. To prove that f is 1-1 we need to show that for two distinct values $x \neq x'$ we have

$$4x+3 \neq 4x'+3. \blacksquare$$

5. Ignore the redundant fourth example.

(1) $f(x) = \frac{x+1}{x} = 1 + \frac{1}{x}$ $(x \neq 0)$. This is 1-1, for $f(a) = f(b) \Rightarrow 1 + \frac{1}{a} = 1 + \frac{1}{b} \Rightarrow \frac{1}{a} = \frac{1}{b} \Rightarrow a = b. \blacksquare$

(2) $f(x) = \frac{x}{x^2+1}$, all $x \in \mathbb{R}$. One way to analyze this is to graph the function. Note that $f(0) = 0$, $\lim_{x \to \infty} f(x) = 0$, and the graph is not strictly increasing. Let's find some specific $a \neq b$ so that $f(a) \neq f(b)$. Take $f(x) = \frac{1}{3}$,

so $\frac{x}{x^2+1} = \frac{1}{3}$, or $x^2 - 3x + 1 = 0$. By the Quadratic Formula, $x = \frac{3 \pm \sqrt{5}}{2}$. Hence f is not 1-1. $\blacksquare$

5. (cont.)

(3) $f(x) = \dfrac{3x-1}{x} = 3 - \dfrac{1}{x}$, $x \neq 0$. In this case $f(a) = f(b) \Rightarrow 3 - \frac{1}{a} = 3 - \frac{1}{b} \Rightarrow a = b$, as before. ∎

6. (a) $f+g$ need not be 1-1. Let $f(x) = x$, $g(x) = -x$. ∎

(b) $f+g$ need not be onto. Use the same examples as in (a). ∎

7. (a) $f \circ g(x) = f(x-1) = (\overset{x-1}{\cancel{x^3}} - 1)^3 = x^3 - 3x^2 + 3x \overset{-}{+} 1$

$g \circ f(x) = g(x^3) = x^3 - 1$. The composites are $\neq$. ∎

(b) $f \circ g(x) = f(x^{1/5}) = (x^{1/5})^5 = x$

$g \circ f(x) = g(x^5) = (x^5)^{1/5} = x$, so composites are $=$. ∎

(c) $g \circ f(n) = g(2n) = [2n/2] = [n] = n$.

$f \circ g(n) = f([\frac{n}{2}]) = 2[\frac{n}{2}]$. So the question is whether $n = 2[\frac{n}{2}]$ always holds. Clearly it fails if n is odd; e.g., if $n = 1$, in which case $2[\frac{n}{2}] = 2 \cdot 0 = 0$. So the composites are $\neq$. ∎

8. The fastest & most elementary way to show the function is 1-1 onto is to proceed as before, rewriting $\frac{x+1}{x-1} = 1 - \frac{2}{x-1}$. This, and the formula for the inverse, will also follow if we solve $y = \frac{x+1}{x-1}$ for x in terms of y. Note that there is no solution to $\frac{x+1}{x-1} = 1$, for then we would have $x+1 = x-1$, so that $2 = 0$.

Now $\frac{x+1}{x-1} = y \Rightarrow x+1 = yx - y \Rightarrow$ $yx - x = y+1$, and since $y \neq 1$ we can solve to get $x = \frac{y+1}{y-1}$. $\blacksquare$

9. Solve $y = 3x+2$ for x. $y-2 = 3x$, so

$$x = \frac{y-2}{3}. \blacksquare$$

10. We know $\cos x$ is 1-1 and nonnegative on $A = [0, \frac{\pi}{2}]$, so the same is true for $\frac{1}{3+\cos^4 x}$. Now the image of the latter is

$B = [\frac{1}{4}, \frac{1}{3}]$ because $0 \leq \cos^4 x \leq 1$ for all x. ■

11. First identity Suppose that $y \in f[A]$, and write $y = f(a)$ where $a \in A$. Then $a \in f^{-1}[f[A]]$, so $y = f(a) \in f[f^{-1}[f[A]]]$.

Conversely, suppose that $y \in f[f^{-1}[f[A]]]$. Since ~~$f^{-1}[f[A]]$~~ $f[f^{-1}[B]] \subseteq B$ always, if we let $B = f[A]$ we get the other containment. ■

Second identity The inclusion in the preceding sentence implies $f^{-1}[f[f^{-1}[B]]] \subseteq f^{-1}[B]$. Conversely, suppose $x \in f^{-1}[B]$, so that $f(x) \in B$, and in fact $f(x) \in f[f^{-1}[B]]$. Then by definition we have $x \in f^{-1}[f[f^{-1}[B]]]$ proving the reverse containment. ■

12. For all x we have $[(f+g)\circ h](x) = [f+g](h(x)) = f(h(x)) + g(h(x)) = f\circ h(x) + g\circ h(x) = [(f\circ h) + (g\circ h)](x)$, verifying the identity. For the nonidentity, let $g(x) = h(x) = x$, $f(y) = y^2$. Then $[f\circ(g+h)](x) = f([g+h](x)) = f(2x) = 4x^2$, but $[f\circ g + f\circ h](x) = 2x^2$. ∎

13. Let $f(x) = x^5$, $g(x) = x^{1/5}$ as in a previous exercise. Additional examples are given by $f(x) = x^{2m+1}$, $\cancel{g(x) = \sqrt[2m+1]{x}}$ ~~w~~ $g(x) = x^{1/(2m+1)}$ where m is an arbitrary positive integer. ∎

TYPO ALERT: $K\cdot(f\circ g)$ should be $C\cdot(f\circ g)$.

14. For all x we have $[C\cdot(f\circ g)](x) =$
$C\cdot f\circ g(x) = C\cdot f(g(x)) = (C\cdot f)\circ g(x)$.
For the counterexample with $f\circ(C\cdot g)$
$\neq C\cdot(f\circ g)$ we can take $C=2$, $f(y)=y^2$,
$g(x)=x$. Then $f\circ(C_g)$ sends x to C^2x^2
but $C\circ(f\circ g)$ sends x to Cx^2. $\blacksquare$