Quiz 3 preparation

1. We begin by restating the identity to be verified:

$$\sum_{k=1}^{n} \frac{2}{k(k+2} \quad = \quad \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$$

If $n = 1$ the left hand side equals $\frac{2}{3}$ and the right hand side equals $\frac{3}{2} - \frac{5}{6} = \frac{9}{6} - \frac{5}{6} = \frac{4}{6} = \frac{2}{3}$, so the identity is true if $n = 1$.

We now need to show that if the identity is true for $n = m \geq 1$ then it is also true for $m + 1$. The starting point is writing out the left hand side when $n = m + 1$:

$$\sum_{k=1}^{m+1} \frac{2}{k(k+2} \quad = \quad \left(\sum_{k=1}^{m} \frac{2}{k(k+2} \right) + \frac{2}{(m+1)(m+3)}$$

Since the identity is known for $n = m$ we may rewrite the right hand side as follows:

$$\frac{3}{2} - \frac{2m+3}{(m+1)(m+2)} + \frac{2}{(m+1)(m+3)}$$

Now we have to combine and simplify the second and third terms, and if we do so here is what we obtain:

$$-\frac{(2m+3)(m+3)}{(m+1)(m+2)(m+3)} + \frac{2(m+2)}{(m+1)(m+2)(m+3)} \quad =$$

$$-\frac{(2m^2+9m+9)-(2m+4)}{(m+1)(m+2)(m+3)} \quad = \quad -\frac{2m+5}{(m+2)(m+3)}$$

The last step follows because $2m^2 + 7m + 5 = (m+1)(2m+5)$.■

2. We begin again by restating the identity to be verified:

$$\prod_{k=1}^{n} \frac{k}{k+1} \quad = \quad \frac{1}{n+1}$$

If $n = 1$ the product on the left hand side consists of a single term, and direct inspection shows that both sides are equal to $\frac{1}{2}$. We now need to show that if the identity is true for $n = m \geq 1$ then it is also true for $m + 1$. But then we have

$$\prod_{k=1}^{m+1} \frac{k}{k+1} \quad = \quad \left(\prod_{k=1}^{m} \frac{k}{k+1} \right) \cdot \frac{m+1}{m+2}$$

and since the identity is known for $n = m$ we may rewrite the right hand side as follows:

$$\frac{1}{m+1} \cdot \frac{m+1}{m+2} \quad = \quad \frac{m+1}{m+2}$$

This completes the proof of the inductive step (namely, if the identity is true for $nn = m$ it is also true for $n = m + 1$).■

3. Once more we begin by restating the identity to be verified:

$$\sum_{k=1}^{n} \frac{k}{2^k} \quad = \quad 2 \; - \; \frac{n+2}{2^n}$$

We can directly verify this identity when $n = 1$ because the left hand side collapses to a single term equal to $\frac{1}{2}$ and the right hand side simplifies to $2 - \frac{3}{2} = \frac{1}{2}$. As before, we now need to show that if the identity is true for $n = m \geq 1$ then it is also true for $m + 1$, and the first step is to split the left hand side into two pieces:

$$\sum_{k=1}^{m+1} \frac{k}{2^k} \quad = \quad \sum_{k=1}^{m} \frac{k}{2^k} \; + \; \frac{m+1}{2^{m+1}}$$

Since the identity is known for $n = m$ we may rewrite the right hand side as follows:

$$2 \; - \; \frac{m+2}{2^m} + \frac{m+1}{2^{m+1}}$$

Finally, we can rewrite the second and third terms in this expression in the form

$$-\;\frac{2m+4}{2^{m+1}} + \frac{m+1}{2^{m+1}} \quad = \quad -\;\frac{(2m+4)-(m+1)}{2^{m+1}} \quad = \quad -\;\frac{m+3}{2^{m+1}}$$

and this shows that if the identity is valid for $n = m$ it is also valid for $n = m + 1$.■

4. By now the pattern should be clear. We begin by restating the identity to be verified:

$$\prod_{k=1}^{n} 4k - 2 \quad = \quad \frac{(2n)!}{n!}$$

If $n = 1$ then both sides simplify to 2, so the identity is true in that case.

We now need to show that if the identity is true for $n = m \geq 1$ then it is also true for $m + 1$. The first step is to rewrite the left hand side:

$$\prod_{k=1}^{m+1} 4k - 2 \quad = \quad \left(\prod_{k=1}^{m} 4k - 2 \right) \cdot (4m + 2)$$

Since the identity is known for $n = m$ the right hand side is equal to

$$\left(\frac{(2m)!}{m!} \right) \cdot (4m + 2) \; .$$

In order to complete the inductive step, we need to show that the latter is equal to

$$\frac{(2m+2)!}{(m+1)!} \quad = \quad \left(\frac{(2m)!}{m!} \right) \cdot \frac{(2m+1)\cdot(2m+2)}{m+1} \; .$$

However, this follows because the last fraction on the right hand side simplifies to $4m + 2$, so we have shown that if the identity is valid for $n = m$ then it is also valid for $n = m + 1$.■