volume of this structure is given by the formula

$$\forall\, n \geq 1, \quad \sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

from Theorem 4.2(d).

(a) Prove that result here by using induction.

(b) How many cubes are required to make the volume of the structure exceed the volume of a single cube with side length 10 inches?

(c) How many cubes are required to make the structure at least 6 feet tall?

In Exercises 5 through 22, do not use Theorems 4.2 and 4.3. Instead, give a proof by induction.

5. Show: $\forall\, n \geq 1, \quad \sum_{i=1}^{n}(3i^2 - i) = n^2(n+1)$.

6. Show: $\forall\, n \geq 1, \quad \sum_{i=1}^{n}(4i^3 - 2i) = n(n+1)(n^2+n-1)$.

7. Show: $\forall\, n \geq 1, \quad 1 + 5 + 9 + \cdots + (4n-3) = n(2n-1)$.

8. Show: $\forall\, n \geq 1, \quad 1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$.

9. Show: $\forall\, n \geq 1, \quad 3 + 5 + 7 + \cdots + (2n+1) = n(n+2)$.

10. Consider the sequence of binary numbers

$$10, 1010, 101010, \ldots$$

whose digits alternate between 1 and 0. If we start our indexing at 0, then the value of the nth number in this sequence is given by

$$2^{2n+1} + \cdots + 2^5 + 2^3 + 2.$$

Show: $\forall\, n \geq 0, \quad 2 + 2^3 + 2^5 + \cdots + 2^{2n+1} = \frac{2}{3}(4^{n+1} - 1)$.

11. A binary tournament, such as the type used for the NCAA basketball tournament held each March, involves a number of teams t that is a power of 2, say $t = 2^{n+1}$. In the first round, the t teams are paired off into 2^n games, and only the winners advance to the second round. There are then 2^{n-1} games in the second round, 2^{n-2} games in the third round, and so forth, until the one game in the final round determines the champion. The total number of games played in this tournament is thus $2^n + 2^{n-1} + 2^{n-2} + \cdots + 1$. In the case that $t = 64 = 2^{5+1}$, the number of games played is $32 + 16 + 8 + 4 + 2 + 1 = 63$. In general, show: $\forall\, n \geq 0, \quad 1 + 2 + 4 + 8 + \cdots + 2^n = 2^{n+1} - 1$.

12. Show: $\forall\, n \geq 0, \quad 1 + 3 + 9 + 27 + \cdots + 3^n = \frac{1}{2}(3^{n+1} - 1)$.

13. Show: $\forall\, n \geq 2, \quad \sum_{i=2}^{n} i2^i = (n-1)2^{n+1}$.

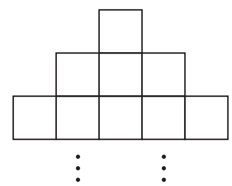
14. Show: $\forall\, n \geq 1, \quad \sum_{i=1}^{n} i3^i = \dfrac{3}{4}[(2n-1)3^n + 1]$.

15. Show: $\forall\, n \geq 1, \quad \sum_{i=1}^{n} i^2 2^i = (n^2 - 2n + 3)2^{n+1} - 6$.

16. Show: $\forall\, n \geq 1,\ \sum_{i=1}^{n} i^2 3^i = \frac{3}{2}[(n^2 - n + 1)3^n - 1]$.

17. Show: $\forall\, n \geq 1,\ \sum_{i=1}^{n} (i \cdot i!) = (n+1)! - 1$.

18. A landscaper wants to have rows of bricks emanating from the backdoor of a house in the following pattern.



By adding the number of bricks in each row, we see that the total number of bricks needed to make n rows in this arrangement is $\sum_{i=1}^{n}(2i-1)$.

Show: $\forall\, n \geq 1,\ \sum_{i=1}^{n} (2i - 1) = n^2$.

19. Show: $\forall\, n \geq 1,\ \sum_{i=1}^{2n} i = n(2n+1)$.

20. Show: $\forall\, n \geq 1,\ \sum_{i=1}^{2n} i^3 = n^2(2n+1)^2$.

21. Show: $\forall\, n \geq 1,\ \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

22. Show: $\forall\, n \geq 1,\ \sum_{i=1}^{n} \frac{2}{i(i+2)} = \frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}$.

23. If a fair coin is tossed i times, then the probability that the first occurrence of "heads" is on the ith toss is $\frac{1}{2^i}$. This is true because on each of the first $i-1$ tosses, there is a probability of $\frac{1}{2}$ that the coin will be "tails," and on the ith toss, there is a probability of $\frac{1}{2}$ that the coin will be "heads." Consequently, if the coin is tossed n times, then the probability that there is some occurrence of heads is $\sum_{i=1}^{n} \frac{1}{2^i}$.

Show: $\forall\, n \geq 1,\ \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$.

24. Show: $\forall\, n \geq 0,\ \sum_{i=0}^{n} (3^{i+1} - 3^i) = 3^{n+1} - 1$.

25. Show: $\forall\, n \geq 1,\ \prod_{i=1}^{n} \frac{i}{i+2} = \frac{2}{(n+1)(n+2)}$.

26. Show: $\forall\, n \geq 1,\ \prod_{i=1}^{n} \frac{i}{i+1} = \frac{1}{n+1}$.

27. Let $r \in \mathbb{R}$. Show: $\forall\, n \geq 1,\ \prod_{i=1}^{n} r^{2i} = r^{n(n+1)}$.

28. Let $m \in \mathbb{Z}$. Show: $\forall\, n \geq 1,\ \prod_{i=1}^{n} i^m = (n!)^m$.