Exercises for Unit I (General considerations**)**

I.1 : Overview of the course

(Halmos**,** *Preface***;** Lipschutz**,** *Preface***)**

Questions to answer:

1. Why are there safety codes mandating that the foundation of a building must meet certain structural requirements? How might one apply the same principle to the mathematical sciences?

2. Why do manufacturers invest substantial resources into studying their mass production methods? How might one apply the same principle to the mathematical sciences?

3. Emphasis on quality standards can sometimes be taken too far. For example, in a chemical laboratory one could focus on cleaning laboratory equipment to a point that it would interfere with performing the experiments that are supposed to be carried out. How might one apply the same principle to the mathematical sciences?

4. The notes mentioned that one reason for the continued study of axiomatic set theory is to test the limits to which the foundations of mathematics can be pushed. Give one or more examples outside of the mathematical sciences where manufacturers might be expected to test the limits of their products.

I.2 : Historical background and motivation

Questions to answer:

1. Zeno's paradoxes are based on mixing "atomic" and "continuous" models for physical phenomena too casually. The following example shows that the casual use of mixed models still happens today: Consider the problem of finding the center of mass for an object like a solid hemisphere by means of calculus. Using calculus to find the center of mass tacitly assumes that matter is continuous. On the other hand, the atomic theory of matter implies that matter is not infinitely divisible. In view of this discrepancy, what meaning should be attached to the integral formula for the center of mass for the solid hemisphere?

2. Find the gap in the following proof that the angle sum **S** of a triangle is always 180 degrees: Let **A, B, C** be the vertices of the triangle, and let **D** be a point between **B** and **C** so that Δ**ABC** is split into Δ**ABD** and Δ**ADC.** The sums of the angles in both triangles are easily computed to be equal to the sums of the angles in Δ**ABC** plus 180 degrees. But since the angle sum of a triangle is **S,** this sum is also equal to 2 **S** and hence we have 2 **S = S +** 180**,** which implies that **S** must be equal to 180. Where is the mistake in this argument? **—** This issue is relevant to the discovery of non – Euclidean geometry because Euclid's Fifth Postulate is equivalent to assuming that the angle sum of a triangle is always 180 degrees. **[*Comment*:** It will probably be helpful to draw a picture corresponding to the assertions made in the fallacious proof.**]**

3. The following example in coordinate geometry illustrates the need for adding assumptions to those in Euclid's ***Elements.*** Consider the triangle in the coordinate plane with vertices **(**0**,** 1**), (**0**,** 0**)** and **(**1**,** 0**),** and let **L** be a line passing through the midpoint of the hypotenuse, which is **(½, ½).** Show that **L** either goes through the vertex **(**0**,** 0**)** or else contains a point on one of the other two sides. It might be helpful to break this problem up into cases depending upon the slope **m** of the line, which might be equal to 1, greater than 1, undefined, less than **–** 1, or between **–** 1 and **+** 1**. —** This example reflects a general fact about triangles and lines meeting one of the sides between the two endpoints, and it is known as ***Pasch's Postulate*** after M. Pasch (1843 – 1930), who noted its significance. This statement is used in the ***Elements,*** but it is not proven and the need to assume it is not acknowledged. **[Comment:** This is not meant to denigrate the ***Elements*;** the purpose is to illustrate that the sorts of problems in Euclid's important and monumental work that were discovered in the late 19th century.**]**

4. Two ordinary decks of 52 cards are shuffled, and each of 52 players is dealt one card from each deck. Explain why at least half of the players will receive two number cards (Ace through 10). **[Hint:** It might be good to break things down into four cases, depending on what sort of card an individual receives from each deck. The number of persons receiving either a number or picture card from a fixed is of course 52, and the number of persons receiving a number card from the first or second deck is 40. Of course, the number of persons in each of the four cases is also nonnegative.**]**

5. The following example shows that some care must be taken when rearranging the terms of an infinite series because different arrangements of the terms sometimes lead to different answers. Consider the standard infinite series for the natural logarithm of 2

$$\ln 2 = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$$

and suppose we rearrange the terms as follows**:**

$$\left(1-\frac{1}{2}\right)-\frac{1}{4}+\left(\frac{1}{3}-\frac{1}{6}\right)-\frac{1}{8}+\left(\frac{1}{5}-\frac{1}{10}\right)-\frac{1}{12}+\ldots$$

Explain why the sum of the rearranged series is **½** $\ln 2$**. —** The general rearrangement question is discussed in pages 75 – 78 of Rudin, ***Principles of Mathematical Analysis*** (3rd Ed., McGraw – Hill, New York, 1976, ISBN: 0–07–054235–X); the subsequent proof of Theorem 3.50 is also relevant to this topic, and additional examples involving the effect of rearrangement on infinite series appear on pages 73 – 74 of Rudin. There is also a discussion with examples on pages 656 – 657 and 660 of the 10th Edition of

Thomas' ***Calculus***, for which bibliographic information is given in the notes. Two points especially worth noting are that the sum of a series of ***nonnegative*** terms does not change if one rearranges the terms in any manner whatsoever and the sum of any series does not change if we only rearrange finitely many terms. More generally, the sum does not change if the series is absolutely convergent (*i.e.*, the series whose terms are the absolute values of the given ones converges). Of course, the latter fails for the series considered above.

6. One important fact about power series is that they can be differentiated term by term, and the result will be the derivative of the function represented by the series. The following example shows that term–by–term differentiation of trigonometric series is not possible. Consider the expansion for the square wave function described in the notes, and consider the termwise second derivatives of both sides away from the points of discontinuity. If it were possible to perform term–by–term differentiation on this series, then one would expect that that the second derivative of the left hand side, which is zero, should be equal to the following infinite series:

$$-4/\pi\,(\sin x + 3\sin 3x + 5\sin 5x + \dots)$$

What happens to this series if, say, $x = \pi/2$?

I.3 : Selected problems

<u>*Questions to answer:*</u>

1. In the setting of the pigeonhole principle, is it possible to place conditions on **m** and **n** which guarantee that at least two locations will contain more than one element? Explain the reasons why or why not. Does the answer change if we put an upper bound on the number of objects that can be placed in a given location? Think about the case $m = 2n$ with a stipulation that no location should contain 3 or more elements.

2. Verify that the base 8 expansion of 1/3 is 0.2525252525 ... by geometric series computations. Recall that in base 8 the expression $0.x_1 x_2 x_3 \dots$ (where each x_k is an integer between 0 and 7) is the sum of the quantities $x_k 8^{-k}$.

3. If **x** is the unique positive cube root of 2, verify that $y = 1 + x$ is an algebraic number by expressing $y^3 = p\,y^2 + q\,y + r$ for suitable integers **p, q** and **r.**

Exercises for Unit II (Basic concepts**)**

II.0 : Topics from logic

(Lipschutz, §§ 10.1 – 10.12**)**

General remarks. This section of the notes has been included mainly as background, and consequently *the exercises below are optional*. However, it probably would be useful to look through some of these items at least briefly. Logical skills play a particularly important role in this course, so a review and assessment of them is extremely worthwhile. The exercises also provide opportunities for ensuring that these skills will be sufficient for the course. Most of the exercises on the list were selected because of their significance for the sorts of proofs one encounters in a course on set theory rather than for the sake of studying symbolic logic in its own right (which is worthwhile, but outside the objectives of this course). The first group of exercises comes from the Discrete Mathematics text by Rosen, which also contains a very large number of other excellent and relevant exercises.

Problems for study.

Lipschutz : 10.8 – 10.10, 10.17 – 10.19, 10.21, 10.23 – 10.28, 10.30

Exercises to work.

Exercises from Rosen:

> ***Note.*** The first two exercises show that one can define the three standard propositional operators *and*, *or* and *not* in terms of a single operator (***Sheffer's stroke***, named after H. M. Sheffer (1883 – 1964), which is written **p|q** and means that at least one of **p** and **q** is false. Biographical information about Sheffer appears on page 26 of Rosen.

1. (Rosen, Exercise 41, p. 27) Show that **p|q** is equivalent to $\neg (p \wedge q)$.

2. (Rosen, Exercise 46, p. 27) Show that all the logical operations $\wedge$, $\vee$, $\neg$ can all be written in terms of Sheffer's stroke. [***Hint:*** Note that $\neg p$ is equivalent to **p|p**, while $p \wedge q$ is equivalent to $\neg (p|q)$, and $p \vee q$ is equivalent to $\neg ([\neg p] \wedge [\neg q])$.]

3. (Rosen, Exercise 41, p. 43) Are the predicate statements $(\forall x)\ [P(x) \Rightarrow Q(x)]$ and $[\forall x, P(x)] \Rightarrow Q(x)$ logically equivalent? Explain why or give a counterexample.

4. (Rosen, Exercise 43, p. 43) Are the predicate statements **$(\exists x)\ [P(x) \vee Q(x)]$** and **$[\exists x, P(x)] \vee [\exists x, Q(x)]$** logically equivalent? Explain why or give a counterexample.

5. (Rosen, Exercise 46, p. 43) Show that the predicate statements

$$(\forall x)\ [P(x) \vee Q(x)] \quad \text{and} \quad [\forall x, P(x)] \vee [\forall x, Q(x)]$$

are not logically equivalent.

6. (Rosen, Exercise 47, p. 43) Show that the predicate statements

$$(\exists x)\ [P(x) \wedge Q(x)] \quad \text{and} \quad [\exists x, P(x)] \wedge [\exists x, Q(x)]$$

are not logically equivalent.

7. (Rosen, Exercise 4, p. 51) Let **$P(x, y)$** be the statement, **"***if* ***x*** *is a student and* ***y*** *is a class, then* ***x*** *has taken class* ***y*.**" Express the following statements in everyday language
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- **(a)** $\exists x\ \exists y\ P(x, y)$
- **(b)** $\exists x\ \forall y\ P(x, y)$
- **(c)** $\forall x\ \exists y\ P(x, y)$
- **(d)** $\exists y\ \forall x\ P(x, y)$
- **(e)** $\forall y\ \exists x\ P(x, y)$
- **(f)** $\forall x\ \forall y\ P(x, y)$

8. (Rosen, Exercise 24, p. 54) Translate each of the following mathematical statements into everyday language:
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- **(a)** $\exists x\ \forall y\ (x + y = y)$
- **(b)** $\forall x\ \forall y\ \left(x \geq 0 \wedge y < 0 \Rightarrow x - y \geq 0\right)$
- **(c)** $\forall x\ \forall y\ \left([x \neq 0] \wedge [y \neq 0] \Leftrightarrow xy \neq 0\right)$
- **(d)** $\exists x\ \exists y\ \left([x^2 \geq y] \wedge [x < y]\right)$
- **(e)** $\forall x\ \forall y\ \exists z\ (x + y = z)$
- **(f)** $\forall x\ \forall y\ \left([x < 0] \wedge [y < 0] \Leftrightarrow xy > 0\right)$

9. (Rosen, Exercise 50, p. 76) Prove that either $2 \cdot 10^{500}$ or $2 \cdot 10^{500} + 1$ is not a perfect square. Is the proof constructive (does it say that a specific choice is not a perfect square) or it nonconstructive?

10. (Rosen, Exercise 51, p. 76) Prove that there is a pair of consecutive integers such that one is a perfect square and the other is a perfect cube.

11. (Rosen, Exercise 52, p. 76) Prove that the product of two of the numbers

$$65 \cdot 10^{1000} - 8^{2001} + 3^{177}, \quad 79^{1212} - 9^{2399} + 2^{2001}, \quad 24^{4493} - 5^{8192} + 7^{1777}$$

is nonnegative without evaluating any of these numbers. Is the proof constructive or nonconstructive?

12. (Rosen, Exercise 57, p. 76) Prove that if n is an odd integer, then there is a unique integer k such that n is the sum of $k - 2$ and $k + 3$.

13. (Rosen, Exercise 11, p. 115) Let **P**(m, n) be the statement that if m and n are positive integers then m (evenly) divides n. Determine which of the following statements are true.
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- **(a)** **P**$(4, 5)$
- **(b)** **P**$(2, 4)$
- **(c)** $\forall m \, \forall n$ **P**(m, n)
- **(d)** $\exists m \, \forall n$ **P**(m, n)
- **(e)** $\exists n \, \forall m$ **P**(m, n)
- **(f)** $\forall n$ **P**$(1, n)$

14. (Rosen, Exercise 26, p. 116) Let x be a real number. Prove that x^3 is an irrational real number, then x is also irrational.

15. (Rosen, Exercise 30, p. 116) Prove that there is a positive integer which can be written as a sum of squares of two other positive integers in more than one way. **[** ***Hint:*** Start with the Pythagorean triples **3, 4, 5** and **5, 12, 13.]**

16. (Rosen, Exercise 31, p. 116) Disprove the statement that every positive integer is the sum of the cubes of eight nonnegative integers. **[** ***Note:*** A result due to J. – L. Lagrange states that every positive integer is the sum of the squares of four integers**.]**

17. (Rosen, Exercise 32, p. 116) Disprove the statement that every positive integer is the sum of at most two squares and a cube of nonnegative integers.

18. (Rosen, Exercise 47, p. 225) Prove or disprove that if you have an **8** gallon jug of water and you have empty jugs with capacities of **5** and **3** gallons respectively, then you can measure four gallons of water by successively pouring the contents of one jug into another jug.

Additional exercises:

1. Suppose that **P, Q** and **R** are logical statements such that **(P $\vee$ R) $\Leftrightarrow$ (Q $\vee$ R).** Give a counterexample to show that the latter does not necessarily imply **P $\Leftrightarrow$ Q.** Also, give a counterexample to show that the analogous condition **(P $\wedge$ R) $\Leftrightarrow$ (Q $\wedge$ R)** does not necessarily imply **P $\Leftrightarrow$ Q.**

2. Suppose that **Q(x, y)** is the following predicate statement:

*If **x** and **y** are odd positive integers, then y^x is a perfect square.*

Determine whether each of the statements $\exists x\ \forall y\ Q(x, y)$ and $\forall x\ \exists y\ Q(x, y)$ is true or false. How does the answer change if the word "odd" is removed?

> **RECALL:** A positive integer $p > 1$ is said to be *prime* if the only possible factorizations of **p** into a product **a b** of positive integers are given by $a = 1$ and $b = p$, or $b = 1$ and $a = p$.

3. Find a counterexample to show that the following conjecture is not true: Every positive integer can be expressed as a sum $p + a^2$, where $a \geq 0$ is an integer and **p** is either a prime or equal to 1.

4. Prove that the only prime number of the form $n^3 - 1$ is given by 7. **[*Hint*:** Look at the usual factorization for the difference of two cubes.**]**

5. Prove that the only prime number **p** such that $3p + 1$ is a perfect square is given by 5. **[*Hint*:** Look at the usual factorization for the difference of two squares.**]**

II.1 : Notation and first steps

(Halmos, § 1**;** Lipschutz, §§ 1.2 – 1.5, 1.10**)**

Questions to answer.

1. Give nonmathematical counterexamples to show that the following statements about set – theoretic membership are not necessarily true for arbitrary sets **A, B, C.**

- ***(i)*** $A \in A$.
- ***(ii)*** If $A \in B$, then $B \in A$.
- ***(iii)*** If $A \in B$ and $B \in C$, then $A \in C$.

2. Give a nonmathematical example of sets **A, B, C** such that $A \subset B$ and $B \in C$ but $A \notin C$.

3. In the set – theoretic approach to classical geometry, space is a set and the points are the elements of that set. Each line or plane will correspond to a subset of space. How might one interpret the concept of a line lying on a plane?

Problems for study.

Lipschutz : 1.1

II.2 : Simple examples

(Halmos, §§ 1 – 3**;** Lipschutz, § 1.12**)**

Problems for study.

Lipschutz : 1.6, 1.41, 1.43(*bcd*), 1.44

Exercises to work.

1. Suppose **A, B** and **C** are sets such that $\mathbf{A} \subset \mathbf{B} \subset \mathbf{C} \subset \mathbf{A}$. Prove that $\mathbf{A} = \mathbf{B} = \mathbf{C}$.

2. Suppose **A, B** and **C** are sets such that **A** is properly contained in **B** and **B** is properly contained in **C.** Prove that **A** is properly contained in **C.**

3. On page 10 of Halmos, the following sets are listed

$$\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \{\varnothing, \{\varnothing\}\}$$

and an exercise in the middle of the page asks if these sets are distinct. Determine whether this is true and give reasons for your answer.

4. Is it possible to find set **A** and **B** such that both $\mathbf{A} \in \mathbf{B}$ and $\mathbf{A} \subset \mathbf{B}$ are true? Give an example or prove this is impossible.

Exercises for Unit III (Elementary constructions on sets**)**

III.1 : Boolean operations

(Halmos, §§ 4 – 5**;** Lipschutz, §§ 1.6 – 1.7**)**

Problems for study.

Lipschutz : 1.9, 1.12 **[*misprint:*** In the first sentence, the expression $\mathbf{B} \cap \mathbf{A}$ should be replaced by $\mathbf{B} \cap \mathbf{A}^{\mathbf{c}}$ **]**, 1.18 – 1.19, 1.46 – 1.47, 1.48 (*gh*), 1.50, 1.52, 1.54 – 1.55, 11.14

Exercises to work.

1. Give examples of sets **A, B, C** such that each of the three pairwise intersections $\mathbf{A} \cap \mathbf{B}$, $\mathbf{A} \cap \mathbf{C}$, $\mathbf{B} \cap \mathbf{C}$ is nonempty but the total intersection $\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}$ is empty (this illustrates the importance of recognizing the difference between a disjoint collection of sets and a pairwise disjoint collection of sets).

2. Is it possible to find a non – disjoint collection of sets **A, B, C** such that at least one pairwise intersection is empty? Prove this or give a counterexample.

3. Prove that unions and intersections satisfy the pair of identities $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{A}) = \mathbf{A}$ and $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{A}) = \mathbf{A}$ (these are called the ***absorption laws***).

4. (Rosen, Exercise 18, p. 95) Let **A, B, C** be subsets of some fixed set **S.** Prove that

$$(\mathbf{A} - \mathbf{B}) - \mathbf{C} = (\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C}).$$

5. (Rosen, Exercise 21, p. 95) What can one say about the sets **A** and **B** if we know the following?

(a) $\mathbf{A} \cup \mathbf{B} = \mathbf{A}$.
(b) $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$.
(c) $\mathbf{A} - \mathbf{B} = \mathbf{A}$.
(d) $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$.
(e) $\mathbf{A} - \mathbf{B} = \mathbf{B} - \mathbf{A}$.

6. Prove or give a counterexample for the following statement: If **A, B, C** are subsets of some set **S** and $\mathbf{A} \cup \mathbf{C} = \mathbf{B} \cup \mathbf{C}$, then $\mathbf{A} = \mathbf{B}$.

7. Prove or give a counterexample for the following statement: If **A** and **B** are subsets of some set **S** and $\mathbf{A} \cap \mathbf{B} = \mathbf{A} \cup \mathbf{B}$, then $\mathbf{A} = \mathbf{B}$.

8. (Rosen, Exercise 22, p. 95) Can one conclude that **A = B** if **A** and **B** are subsets of some set **S** and satisfy the following? There are actually **<u>three</u>** parts to this problem, corresponding to whether **(1), (2),** or both are known to be true. Give reasons for your answer in all three cases.

> **(1)** $\mathbf{A \cup C = B \cup C}$.
> **(2)** $\mathbf{A \cap C = B \cap C}$.

9. (Rosen, Exercise 41, p. 116) Let **A, B, C** be subsets of some fixed set **S.** Show that the set $\mathbf{(A - B) - C}$ is not necessarily equal to $\mathbf{A - (B - C)}$.

10. (Rosen, Exercise 43, p. 116) Let **A, B, C, D** be subsets of some fixed set **S.** Prove or disprove that $\mathbf{(A - B) - (C - D) = (A - C) - (B - D)}$.

11. (Rosen, Exercise 14, p. 95) Let **A, B, C** be subsets of some fixed set **S.** Prove that the following hold:

> **(a)** $\mathbf{A \cup B \subset (A \cup B) \cup C}$.
> **(b)** $\mathbf{(A \cap B) \cap C \subset A \cap B}$.
> **(c)** $\mathbf{(A - B) - C \subset A - C}$.
> **(d)** $\mathbf{(A - C) \cap (C - B) = \emptyset}$.
> **(e)** $\mathbf{(B - A) \cup (C - A) = (B \cup C) - A}$.

12. Suppose that we are given sets **A, B, C, D** such that $\mathbf{A \subset C}$ and $\mathbf{B \subset D}$. Prove that $\mathbf{A \cup B \subset C \cup D}$ and $\mathbf{A \cap B \subset C \cap D}$.

13. Suppose that **A** and **B** are subsets of a given set **S.** Prove that $\mathbf{A \subset B}$ if and only if $\mathbf{S - B \subset S - A}$.

14. (Halmos, p. 16) Let **A, B,** and **C** be subsets of a given set **S.** Prove that one has the mixed associativity (also known as ***modularity***) property

$$\mathbf{(A \cap B) \cup C = A \cap (B \cup C)}$$

if and only if $\mathbf{C \subset A}$; in particular, the criterion has nothing to do with **B.**

III.2 : Ordered pairs and products

(Halmos, §§ 3, 6**;** Lipschutz, §§ 3.1 – 3.2**)**

<u>*Problems for study.*</u>

Lipschutz : 3.2 – 3.4, 3.35 – 3.36, 3.39 – 3.40

Exercises to work.

1. Prove the following identities for Cartesian products:

(1) $A \times (B \cap D) = (A \times B) \cap (A \times D)$

(2) $A \times (B \cup D) = (A \times B) \cup (A \times D)$

(3) $A \times (Y - D) = (A \times Y) - (A \times D)$

(4) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(5) $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$

(6) $(X \times Y) - (A \times B) = [X \times (Y - B)] \cup [(X - A) \times Y]$

2. (Part of Halmos, p. 25) Let **A** and **B** be sets. Explain why $A \times B$ is empty if either **A** is empty or **B** is empty.

3. Suppose that **A** and **B** are sets such that $(A \times B) \cap (B \times A)$ is empty. What conclusion can be drawn regarding **A** and **B?**

III.3 : Larger constructions

(Halmos, §§ 3, 5 – 6, 9**;** Lipschutz, §§ 1.9, 3.1 – 3.2, 5.1 – 5.2**)**

Problems for study.

Lipschutz : 1.66(*b*), 3.35, 5.1, 5.3, 5.28, 5.31

Exercises to work.

1. Let **F** be the family of all closed intervals on the real line having the form **[–M, M]**, where a real number **x** belongs to **[–M, M]** if and only if the absolute value of **x** is less than or equal to the positive real number **M.** Describe **$(F)** and $\cap$ **{B | B $\in$ F }.** What happens if instead we look at the family of ***open*** intervals having the form **(–M, M)**, where a real number **x** belongs to **(–M, M)** if and only if the absolute value of **x** is strictly less than the positive real number **M?**

2. For each integer **n > 1** let **L(n)** be the set of all real numbers **x** such that $x^n < x$, and let **F** be the family of all subsets having the form **L(n)** for some **n > 1.** Describe **$(F)** and $\cap$ **{B | B $\in$ F }.**

3. (Halmos, p. 20) Prove that power set construction satisfies the algebraic conditions $P(A) \cap P(B) = P(A \cap B)$ and $P(A) \cup P(B) \subset P(A \cup B)$. More generally, prove that

$$\cap_{A \in C} P(A) = P(\cap_{A \in C} A) \text{ if } C \text{ is nonempty and}$$

$$\cup_{A \in C} P(A) \subset P(\cup_{A \in C} A) \text{ if } C \text{ is arbitrary.}$$

In the case of unions, give an example for which the containment is proper.

4. (Rosen, Exercise 16, p. 85) Can we conclude that **A = B** if **P(A) = P(B)?** Give reasons for your answer.

Defining ordered triples. A method for defining ordered pairs of sets formally is described on page 23 of Halmos (see the bottom half of the page), and the next few pages contain remarks about the definition. Specifically, Halmos defines the ordered pair **(a, b)** to be **{ {a}, {a, b} }.** One can use this to define an ***ordered triple*** **(a, b, c)** in terms of ordered pairs by the rule

$$(a, b, c) = \big((a, b) , c \big).$$

5. Explain why **(a, b, c) = (x, y, z)** if and only if **a = x, b = y** and **c = z.**

6. Prove that the construction $\langle a, b, c \rangle = \{ \{a\}, \{a, b\}, \{a, b, c\} \}$ does ***not*** satisfy the property of ordered triples established in the preceding exercise**.** **[*Hint*:** One can show that the property holds if **a = b = c** or all three of **a, b, c** are distinct, so the search for counterexamples should begin by looking at cases where exactly two of the sets are equal**.]**

III.4 : A convenient assumption

(Halmos, §§ 2**;** Lipschutz, § 1.12**)**

Exercises to work.

1. The Axiom of Foundation implies that one cannot have an infinite sequence of sets x_k (for $k \geq 0$) such that $x_k \in x_{k-1}$ for all $k > 0$. We shall say that a nonempty set **x** has ***Russell type*** equal to **n** if there exists a sequence of the form

$$\mathbf{x_n \in x_{n-1} \in \ldots \in x_1 \in x}$$

but there are no sequences such that $\mathbf{y_{n+1} \in \ldots \in y_1 \in x}$. By definition, the empty set will have Russell type 0. Prove that for each nonnegative integer **n** there is a set with Russell type **n.** **[*<u>Hint</u>*:** If a set **x** has Russell type equal to **k,** what is the Russell type of **{x}?]**

> ***<u>Note.</u>*** One can extend this to say that a set has *<u>infinite Russell type</u>* if for each **n** there is a sequence $\mathbf{x_n \in x_{n-1} \in \ldots \in x_1 \in x}$. On the basis of what we have done thus far, we cannot conclude that such sets exist, but this will follow from the material in Unit **V.**

2. Let us say that a set has *<u>finite Russell type</u>* if it has Russell type **n** for some nonnegative integer **n.** Prove that the union of two sets with finite Russell type also has finite Russell type.

3. Suppose that we ***define*** the ordered pair **(x, y)** to be the set **{ {x}, {x, y} }** as in pages 23 – 25 of Halmos (and elsewhere**!**) rather than assume its existence. Prove that if **A** and **B** are two sets with finite Russell type, then their Cartesian product also has finite Russell type.

4. If the set **A** has finite Russell type, is the same true of the power set **P(A)?** If this is true and the Russell type of **A** is **n,** then what is the Russell type of **P(A)?**

Exercises for Unit III (Elementary constructions on sets**)**

III.1 : Boolean operations

(Halmos, §§ 4 – 5**;** Lipschutz, §§ 1.6 – 1.7**)**

Problems for study.

Lipschutz : 1.9, 1.12 **[** ***misprint:*** In the first sentence, the expression $\mathbf{B} \cap \mathbf{A}$ should be replaced by $\mathbf{B} \cap \mathbf{A}^{\mathbf{c}}$ **]**, 1.18 – 1.19, 1.46 – 1.47, 1.48 (*gh*), 1.50, 1.52, 1.54 – 1.55, 11.14

Exercises to work.

1. Give examples of sets **A, B, C** such that each of the three pairwise intersections $\mathbf{A} \cap \mathbf{B}$, $\mathbf{A} \cap \mathbf{C}$, $\mathbf{B} \cap \mathbf{C}$ is nonempty but the total intersection $\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}$ is empty (this illustrates the importance of recognizing the difference between a disjoint collection of sets and a pairwise disjoint collection of sets).

2. Is it possible to find a non – disjoint collection of sets **A, B, C** such that at least one pairwise intersection is empty? Prove this or give a counterexample.

3. Prove that unions and intersections satisfy the pair of identities $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{A}) = \mathbf{A}$ and $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{A}) = \mathbf{A}$ (these are called the ***absorption laws***).

4. (Rosen, Exercise 18, p. 95) Let **A, B, C** be subsets of some fixed set **S.** Prove that

$$(\mathbf{A} - \mathbf{B}) - \mathbf{C} = (\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C}).$$

5. (Rosen, Exercise 21, p. 95) What can one say about the sets **A** and **B** if we know the following?

- **(a)** $\mathbf{A} \cup \mathbf{B} = \mathbf{A}$.
- **(b)** $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$.
- **(c)** $\mathbf{A} - \mathbf{B} = \mathbf{A}$.
- **(d)** $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$.
- **(e)** $\mathbf{A} - \mathbf{B} = \mathbf{B} - \mathbf{A}$.

6. Prove or give a counterexample for the following statement: If **A, B, C** are subsets of some set **S** and $\mathbf{A} \cup \mathbf{C} = \mathbf{B} \cup \mathbf{C}$, then $\mathbf{A} = \mathbf{B}$.

7. Prove or give a counterexample for the following statement: If **A** and **B** are subsets of some set **S** and $\mathbf{A} \cap \mathbf{B} = \mathbf{A} \cup \mathbf{B}$, then $\mathbf{A} = \mathbf{B}$.

8. (Rosen, Exercise 22, p. 95) Can one conclude that **A = B** if **A** and **B** are subsets of some set **S** and satisfy the following? There are actually <u>**three**</u> parts to this problem, corresponding to whether **(1), (2),** or both are known to be true. Give reasons for your answer in all three cases.

(1) $\mathbf{A \cup C = B \cup C}$.
(2) $\mathbf{A \cap C = B \cap C}$.

9. (Rosen, Exercise 41, p. 116) Let **A, B, C** be subsets of some fixed set **S.** Show that the set $\mathbf{(A - B) - C}$ is not necessarily equal to $\mathbf{A - (B - C)}$.

10. (Rosen, Exercise 43, p. 116) Let **A, B, C, D** be subsets of some fixed set **S.** Prove or disprove that $\mathbf{(A - B) - (C - D) = (A - C) - (B - D)}$.

11. (Rosen, Exercise 14, p. 95) Let **A, B, C** be subsets of some fixed set **S.** Prove that the following hold:

(a) $\mathbf{A \cup B \subset (A \cup B) \cup C}$.
(b) $\mathbf{(A \cap B) \cap C \subset A \cap B}$.
(c) $\mathbf{(A - B) - C \subset A - C}$.
(d) $\mathbf{(A - C) \cap (C - B) = \emptyset}$.
(e) $\mathbf{(B - A) \cup (C - A) = (B \cup C) - A}$.

12. Suppose that we are given sets **A, B, C, D** such that $\mathbf{A \subset C}$ and $\mathbf{B \subset D}$. Prove that $\mathbf{A \cup B \subset C \cup D}$ and $\mathbf{A \cap B \subset C \cap D}$.

13. Suppose that **A** and **B** are subsets of a given set **S.** Prove that $\mathbf{A \subset B}$ if and only if $\mathbf{S - B \subset S - A}$.

14. (Halmos, p. 16) Let **A, B,** and **C** be subsets of a given set **S.** Prove that one has the mixed associativity (also known as ***modularity***) property

$$\mathbf{(A \cap B) \cup C = A \cap (B \cup C)}$$

if and only if $\mathbf{C \subset A}$**;** in particular, the criterion has nothing to do with **B.**

III.2 : Ordered pairs and products

(Halmos, §§ 3, 6**;** Lipschutz, §§ 3.1 – 3.2**)**

<u>*Problems for study.*</u>

Lipschutz : 3.2 – 3.4, 3.35 – 3.36, 3.39 – 3.40

Exercises to work.

1. Prove the following identities for Cartesian products:

(1) $A \times (B \cap D) = (A \times B) \cap (A \times D)$

(2) $A \times (B \cup D) = (A \times B) \cup (A \times D)$

(3) $A \times (Y - D) = (A \times Y) - (A \times D)$

(4) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(5) $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$

(6) $(X \times Y) - (A \times B) = [\, X \times (Y - B)\,] \cup [\,(X - A) \times Y\,]$

2. (Part of Halmos, p. 25) Let **A** and **B** be sets. Explain why $A \times B$ is empty if either **A** is empty or **B** is empty.

3. Suppose that **A** and **B** are sets such that $(A \times B) \cap (B \times A)$ is empty. What conclusion can be drawn regarding **A** and **B?**

III.3 : Larger constructions

(Halmos, §§ 3, 5 – 6, 9**;** Lipschutz, §§ 1.9, 3.1 – 3.2, 5.1 – 5.2**)**

Problems for study.

Lipschutz : 1.66(*b*), 3.35, 5.1, 5.3, 5.28, 5.31

Exercises to work.

1. Let **F** be the family of all closed intervals on the real line having the form **[–M, M]**, where a real number **x** belongs to **[–M, M]** if and only if the absolute value of **x** is less than or equal to the positive real number **M.** Describe **$(F)** and $\cap$ **{B | B $\in$ F }.** What happens if instead we look at the family of ***open*** intervals having the form **(–M, M)**, where a real number **x** belongs to **(–M, M)** if and only if the absolute value of **x** is strictly less than the positive real number **M?**

2. For each integer $\mathbf{n} > \mathbf{1}$ let **L(n)** be the set of all real numbers **x** such that $\mathbf{x^n} < \mathbf{x}$, and let **F** be the family of all subsets having the form **L(n)** for some **n > 1.** Describe **$(F)** and $\cap$ **{B | B $\in$ F }.**

3. (Halmos, p. 20) Prove that power set construction satisfies the algebraic conditions $\mathbf{P(A) \cap P(B) = P(A \cap B)}$ and $\mathbf{P(A) \cup P(B) \subset P(A \cup B)}$. More generally, prove that

$$\cap_{A \in C} \mathbf{P(A)} = \mathbf{P}(\cap_{A \in C} \mathbf{A}) \text{ if } \mathbf{C} \text{ is nonempty and}$$

$$\cup_{A \in C} \mathbf{P(A)} \subset \mathbf{P}(\cup_{A \in C} \mathbf{A}) \text{ if } \mathbf{C} \text{ is arbitrary.}$$

In the case of unions, give an example for which the containment is proper.

4. (Rosen, Exercise 16, p. 85) Can we conclude that **A = B** if **P(A) = P(B)?** Give reasons for your answer.

> ***Defining ordered triples.*** A method for defining ordered pairs of sets formally is described on page 23 of Halmos (see the bottom half of the page), and the next few pages contain remarks about the definition. Specifically, Halmos defines the ordered pair **(a, b)** to be **{ {a}, {a, b} }.** One can use this to define an ***ordered triple*** **(a, b, c)** in terms of ordered pairs by the rule
>
> $$\mathbf{(a, b, c) = ((a, b), c).}$$

5. Explain why **(a, b, c) = (x, y, z)** if and only if **a = x, b = y** and **c = z.**

6. Prove that the construction $\langle$ **a, b, c** $\rangle$ **= { {a}, {a, b}, {a, b, c} }** does ***not*** satisfy the property of ordered triples established in the preceding exercise**.** **[*Hint*:** One can show that the property holds if **a = b = c** or all three of **a, b, c** are distinct, so the search for counterexamples should begin by looking at cases where exactly two of the sets are equal**.]**

III.4 : A convenient assumption

(Halmos, §§ 2**;** Lipschutz, § 1.12**)**

Exercises to work.

1. The Axiom of Foundation implies that one cannot have an infinite sequence of sets $\mathbf{x_k}$ (for $\mathbf{k} \geq 0$) such that $\mathbf{x_k} \in \mathbf{x_{k-1}}$ for all $\mathbf{k} > 0$**.** We shall say that a nonempty set **x** has ***Russell type*** equal to **n** if there exists a sequence of the form

$$x_n \in x_{n-1} \in \ldots \in x_1 \in x$$

but there are no sequences such that $y_{n+1} \in \ldots \in y_1 \in x$. By definition, the empty set will have Russell type 0. Prove that for each nonnegative integer **n** there is a set with Russell type **n.** **[*Hint*:** If a set **x** has Russell type equal to **k,** what is the Russell type of **{x}?]**

> ***Note.*** One can extend this to say that a set has *infinite Russell type* if for each **n** there is a sequence $x_n \in x_{n-1} \in \ldots \in x_1 \in x$. On the basis of what we have done thus far, we cannot conclude that such sets exist, but this will follow from the material in Unit **V.**

2. Let us say that a set has *finite Russell type* if it has Russell type **n** for some nonnegative integer **n.** Prove that the union of two sets with finite Russell type also has finite Russell type.

3. Suppose that we ***define*** the ordered pair **(x, y)** to be the set **{ {x}, {x, y} }** as in pages 23 – 25 of Halmos (and elsewhere**!**) rather than assume its existence. Prove that if **A** and **B** are two sets with finite Russell type, then their Cartesian product also has finite Russell type.

4. If the set **A** has finite Russell type, is the same true of the power set **P(A)?** If this is true and the Russell type of **A** is **n,** then what is the Russell type of **P(A)?**

Exercises for Unit IV (Relations and functions**)**

IV.1 : Binary relations

(Halmos, § 6**;** Lipschutz, §§ 3.3 – 3.9, 3.11, 7.1 – 7.6, 7.8**)**

Problems for study.

Lipschutz : 3.6(*a*), 3.7(*b*), 3.11, 3.12(*b*), 3.13 – 3.14, 3.16 – 3.18, 3.23 – 3.25, 3.29 – 3.30, 3.32 – 3.33, 3.41(*ab*), 3.45 – 3.50, 3.55, 3.57.

Exercises to work.

1. (Rosen, Exercise 3, p. 480) Determine whether each of the following relations on the set $\{1, 2, 3, 4\}$ is reflexive, symmetric, antisymmetric or transitive.

- **(a)** $\{ (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4) \}$
- **(b)** $\{ (1, 1), (2, 2), (2, 1), (1, 2), (3, 3), (4, 4) \}$
- **(c)** $\{ (2, 4), (4, 2) \}$
- **(d)** $\{ (1, 2), (2, 3), (3, 4) \}$
- **(e)** $\{ (1, 1), (2, 2), (3, 3), (3, 4) \}$
- **(f)** $\{ (1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4) \}$

2. (Taken from Rosen, Exercise 6, p. 480) Determine whether the relations described by the conditions below are reflexive, symmetric, antisymmetric or transitive.

- **(c)** All ordered pairs of real numbers (x, y) such that $x - y$ is rational.
- **(d)** All ordered pairs of real numbers (x, y) such that $x = 2y$.
- **(e)** All ordered pairs of real numbers (x, y) such that $xy \geq 0$.
- **(f)** All ordered pairs of real numbers (x, y) such that $xy = 0$.

3. (Taken from Rosen, Exercise 7, p. 480) Determine whether the relations described by the conditions below are reflexive, symmetric, antisymmetric or transitive.

- **(a)** All ordered pairs of real numbers (x, y) such that $x \neq y$.
- **(b)** All ordered pairs of real numbers (x, y) such that $xy \geq 1$.
- **(c)** All ordered pairs of real numbers (x, y) such that $x = y \pm 1$.
- **(g)** All ordered pairs of real numbers (x, y) such that $x = y^2$.
- **(h)** All ordered pairs of real numbers (x, y) such that $x \geq y^2$.

4. (Taken from Rosen, Exercise 7, p. 533) Suppose that R_1 and R_2 are reflexive relations on a set **A.** Are their union and intersection reflexive? Give reasons for your answer.

5. (Taken from Rosen, Exercise 1, p. 513) Which of the relations described below on the set **{0, 1, 2, 3}** are equivalence relations? Determine the properties of an equivalence relation that the others lack.

- **(a)** **{ (0, 0), (1, 1), (2, 2), (3, 3) }**
- **(b)** **{ (0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3) }**
- **(c)** **{ (0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3) }**
- **(d)** **{ (0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) }**
- **(e)** **{ (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3) }**

6. (Taken from Rosen, Exercise 2, p. 513) Which of the relations described below on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

- (a) All **a** and **b** such that **a** and **b** have the same age.
- (b) All **a** and **b** such that **a** and **b** have the same parents.
- (c) All **a** and **b** such that **a** and **b** have a common parent.
- (d) All **a** and **b** such that **a** and **b** have met.
- (e) All **a** and **b** such that **a** and **b** speak a common language.

7. Let ***R*** be a binary relation that is reflexive and transitive, and define a new binary relation ***S*** such that **x *S* y** if and only if **x *R* y** and **y *R* x.** Prove that ***S*** is an equivalence relation.

8. (Taken from Rosen, Exercise 10, p. 533) A relation ***R*** is said to be ***<u>circular</u>*** if it satisfies **a *R* b** and **b *R* c** imply **c *R* a.** Show that ***R*** is an equivalence relation if and only if it is reflexive and circular.

9. Let $\mathbf{R}$ denote the real numbers, and let ***P*** be the binary relation on $\mathbf{R} \times (\mathbf{R} - \{0\})$ such that **(x, y) *P* (z, w)** if and only if **xw = yz.** Prove that ***P*** is an equivalence relation, and show that every equivalence class contains a unique element (or representative) of the form **(r, 1).**

10. Let $\mathbf{N}^+$ be the set of all positive integers, and define a binary relation ***Q*** on the set $\mathbf{N}^+ \times \mathbf{N}^+$ such that **(x, y) *Q* (z, w)** if and only if $x^w = z^y$. Determine whether ***Q*** is an equivalence relation.

11. Let ***R*** be the binary relation in Algebraic Example 3 (the set is a chessboard, and the relation is that two squares are related if there is a knight's move from one to the other).

(i) Let ***E*** be the equivalence relation generated by ***R.*** Show that ***E*** contains exactly one equivalence class; in other words, starting from the standard position of **(1, 2)** the knight can reach every point on the chessboard.

(ii) Suppose that we replace our 8×8 chessboard with an infinite board whose elements are ordered pairs of integers. Prove that in this case the equivalence relation ***E*** generated by ***R*** also has one point. **[*Hint* :** Start at the origin, and show that every adjacent square is ***E* –** related to it.**]**

12. (Taken from Rosen, Exercise 38, p. 481) Let R_1 be the relation **"a** divides **b"** on the positive integers, and let R_2 be the relation **"a** is a multiple of **b"** on positive integers. Describe the relations $R_1 \cup R_2$ and $R_1 \cap R_2$ **.**

13. Given two binary relations ***S*** and ***T*** on a set **A,** their composite $S \circ T$ is defined to be all $(x, z) \in A \times A$ such that there is some $y \in A$ for which **x *S* y** and **y *T* z.** **If A** is the real line and ***S*** and ***T*** are the relations

$$u \, S \, v \text{ if and only if } |u| = |v|$$

$$u \, T \, v \text{ if and only if } |u + 1| = |v - 2|$$

then find all real numbers **z** such that $1 \; S \circ T \; z$ and all real number **z** such that $2 \; S \circ T \; z$**.** **[*Hint:*** the first relation amounts to saying that $v = a u$ where $a = \pm 1$, and the second amounts to saying that $v - 2 = b(u - 1)$ where once again $b = \pm 1$. Why does this imply that there are at most ***four*** choices for **z** in each case**?]**

14. Let ***S*, T_1** and T_2 be binary relations on a set **A.** Prove that we have $S \circ (T_1 \cup T_2) = (S \circ T_1) \cup (S \circ T_2)$ and $S \circ (T_1 \cap T_2) \subset (S \circ T_1) \cap (S \circ T_2)$. Find an example where the inclusion in the second statement is proper. **[*Hint:*** There is an example for which **A** has four elements**.]**

IV.2 : Partial and linear orderings

(Halmos, § 14**;** Lipschutz, §§ 3.10, 7.1 – 7.6**)**

Problems for study.

Lipschutz : 7.1, 7.3 – 7.5, 7.9, 7.11, 7.27(*b*), 7.41 – 7.42, 7.52.

Exercises to work.

1. Let P_1 and P_2 be partial orderings on the set **A.** Prove that $P_1 \cap P_2$ is a partial ordering, and give an example to show that $P_1 \cup P_2$ is not necessarily a partial ordering.

2. (Rosen, Exercise 10, p. 528) Let $S = \{1, 2, 3, 4\}$ with the usual ordering, and take the lexicographic ordering on $S \times S$. Find all elements of $S \times S$ which are less than $(2, 3)$, and find all elements of $S \times S$ which are greater than $(3, 1)$.

3. Let **J** be the set of closed intervals in the real numbers, and define a binary relation ***P*** such that $[a,b] \; P \; [c,d]$ if and only if $[a,b] = [c,d]$ or $b < c$.

(1) Show that ***P*** defines a partial ordering on **J.**

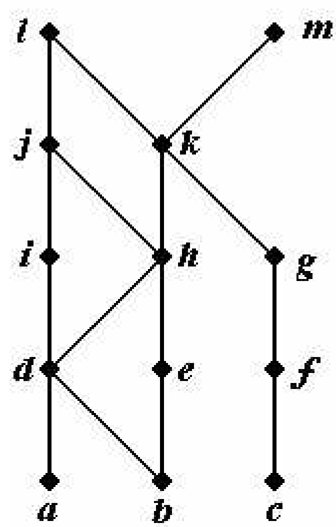
(2) Show that two elements of ***P*** are comparable if and only if they are equal or disjoint.

(3) Show that ***P*** is not a linear ordering on **J.**

4. Let **S** be the set $\{1, \ldots, n\}$. Prove that **P(S)** has a linearly ordered subset **T** with $n + 1$ elements but **S** does not contain a linearly ordered subset with $n + 2$ elements**.**

5. Let **R[t]** be the set of all polynomials with real coefficients, and define $p \leq q$ if and only if $p(x) \leq q(x)$ for all real values of **x.** Prove this defines a partial ordering of **R[t],** but that this partial ordering is not a linear ordering.

6. (Taken from Rosen, Exercise 26, pp. 528 – 529) Answer these questions for the partially ordered set represented by following <u>*Hasse diagram*</u> (the latter is defined with an example on page 170 of Lipschutz):



(a) Find the maximal elements.
(b) Find the minimal elements.
(c) Is there a greatest element?
(d) Is there a least element?
(e) Find all upper bounds of **{a, b, c}**.
(f) Find the least upper bound of **{a, b, c}** if it exists.
(g) Find all lower bounds of **{f, g, h}**.
(h) Find the greatest lower bound of **{f, g, h}** if it exists.

7. Let **(X, ≤)** be a linearly ordered set, and let **a, b, c** be three ***distinct*** elements of **X.** We shall say that **b** <u>*is between*</u> **a** <u>*and*</u> **c** if either $a < b < c$ or $c < b < a$ is true**.** Explain why if **b** is between **a** and **c,** then **b** is also between **c** and **a,** and also prove that if **(X, ≤)** is a linearly ordered set such that **x, y, z** are three distinct elements of **X,** then one and only one of these elements is between the other two.

IV.3 : Functions

(Halmos, §§ 9 – 10**;** Lipschutz, §§ 3.3 – 3.9, 4.1 – 4.4, 5.6, 5.8**)**

Problems for study.

Lipschutz : 4.1 – 4.2, 4.3(*ac*), 4.7, 4.8, 4.33, 4.35, 4.37.

Exercises to work.

1. Let **P** be the set of all U. S. presidents, and let **G** be the set of all ordered pairs **(a, b)** in $\mathbf{P} \times \mathbf{P}$ such that **b** succeeded **a** in office. Is **G** the graph of a function? Explain your answer.

2. Let **A** and **x** be sets. Prove that there is a unique function from **A** to $\{x\}$. It might be helpful to split the proof into two cases depending upon whether or not **A** is empty.

3. (Halmos, p. 33) Prove that for each set **X** there is a unique function from the empty set to **X,** regardless of whether or not **X** is nonempty. Also prove that there are no functions from **X** to the empty set if **X** is nonempty.

4. (Taken from Rosen, Exercise 4, p. 108) Find the domain and range of the function which assigns to each nonnegative integer its last digit.

5. (Taken from Rosen, Exercise 7, p. 109) Find the domain and range of the function which assigns to each positive integer the number of digits **1, 2, 3, 4, 5, 6, 7, 8, 9** that do not appear in the base **10** decimal expansion of the integer.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 3x - 7$. Compute the following sets:

(i)	$f^{-1}[\{3\}]$
(ii)	$f[\{5\}]$
(iii)	$f^{-1}[[-7, 2]]$
(iv)	$f[[2, 6]]$
(v)	$f[\emptyset]$
(vi)	$f^{-1}[[3, 5] \cup [8, 10]]$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = (x + 1)^2$. Compute the following sets:

(i)	$f[\{-1\}]$
(ii)	$f^{-1}[[0,1]]$
(iii)	$f^{-1}[[-1, 1]]$
(iv)	$f[[-1, -1] \cup [1, 3]]$
(v)	$f\left[f^{-1}[[-3, -1]]\right]$
(vi)	$f\left[f^{-1}[[-1, 1]]\right]$

IV.4 : Composite and inverse functions

(Halmos, § 10**;** Lipschutz, §§ 4.3 – 4.4, 5.7**)**

Problems for study.

Lipschutz : 4.11 – 4.14, 4.17 – 4.18, 4.20, 4.22, 4.39(*b*), 4.45 – 4.46.

Exercises to work.

1. Suppose that **X** is linearly ordered set, **Y** is a partially ordered set, and $\mathbf{f : X \rightarrow Y}$ is strictly increasing (***i.e.*, a < b** implies **f(a) < f(b)** for all **a, b**). Prove that **f** is **1 – 1.** Give an example to show that this fails if **X** is not linearly ordered.

2. Suppose that **A** and **B** are sets. Show that the mapping

$$\mathbf{h : P(A) \times P(B) \rightarrow P(A \times B)}$$

which sends **(C, D)** to $\mathbf{C \times D \subset A \times B}$ is **1 – 1,** and give an example to show it is not necessarily onto.

3. (Taken from Rosen, Exercise 10, p. 109) Determine whether each of the following functions from the set **{a, b, c, d}** to itself is injective.

- **(a)** The function sending the ordered quadruple **(a, b, c, d)** to **(b, a, c, d).**
- **(b)** The function sending the ordered quadruple **(a, b, c, d)** to **(b, b, d, c)**.
- **(c)** The function sending the ordered quadruple **(a, b, c, d)** to **(d, b, c, d)**.

4. (Taken from Rosen, Exercise 18, p. 109) Determine which of these functions are bijections from the set of real numbers to itself**.**

- **(a)** $\mathbf{f(x) = -3x + 4.}$
- **(b)** $\mathbf{f(x) = -3x^2 + 7.}$
- **(c)** $\mathbf{f(x) = (x + 1)/(x + 2)}$ **.**
- **(d)** $\mathbf{f(x) = x^5 + 1.}$

5. A function $\mathbf{f : A \rightarrow B}$ is called a ***formal monomorphism*** if for all functions $\mathbf{g, h : C \rightarrow A,}$ the equation $\mathbf{fg = fh}$ implies $\mathbf{g = h.}$ Prove that **f** is a formal monomorphism if and only if **f** is injective.

6. Similarly, a function $\mathbf{f : A \rightarrow B}$ is called a ***formal epimorphism*** if for all functions $\mathbf{g, h : B \rightarrow D,}$ the equation $\mathbf{gf = hf}$ implies $\mathbf{g = h.}$ Prove that **f** is a formal epimorphism if and only if **f** is surjective**.**

7. (Halmos, p. 41) Let **X** and **Y** be nonempty sets, and let $f : X \to Y$ be a function.

- **(a)** Prove that $f(A \cap B) = f(A) \cap f(B)$ for all subsets **A** and **B** of **X** if and only if **f** is $1-1$.
- **(b)** Prove that $f(X - A) \subset Y - f(A)$ for all subsets **A** of **X** if and only if **f** is $1-1$.
- **(c)** Prove that $Y - f(A) \subset f(X - A)$ for all subsets **A** of **X** if and only if **f** is onto.

8. Let **A** and **B** be nonempty sets, and let $f : A \to B$ be a $1-1$ function. Prove that there is a one – sided inverse $g : B \to A$; ***i.e.,*** we have $gf = id_A$. [***Hint:*** Given an element $z \in A$, define **g** as follows: If $b \in B$ can be written as **f(a)** for some **a,** then set $g(b) = a$; this is well – defined because **f** is injective. Otherwise, let $g(b) = z$.]

9. A function $f : A \to B$ is called a ***retract*** if there is a function $g : B \to A$ such that $gf = 1_A$. Prove that every retract is a monomorphism (this is a converse to a previous exercise). Also prove that the associated map **g** is an epimorphism.

10. A function $f : A \to B$ is called a ***retraction*** if there is a function $g : B \to A$ such that $fg = 1_B$. Prove that every retraction is an epimorphism (this is a converse to a previous exercise). Also prove that **g** is a monomorphism.

11. Let $[0, 1]$ be the closed unit interval, and let **a** and **b** be real numbers which satisfy $a < b$. Construct a bijection from $[0, 1]$ to $[a, b]$. Is it unique?

12. Give examples of composable functions **f** and **g** such that **g f** is a bijection but neither **f** nor **g** is a bijection. If **g f** is a bijection, is either of **g** or **f** an injection or a surjection? The preceding question has four separate parts.

13. Find the inverse functions to $p(x) = 3x - 1$ and $q(x) = x/(1 + |x|)$, where the domains of both functions are the real numbers, the codomain of **p** is also the reals, and the codomain of **q** is $(-1, 1)$. [***Hint:*** In the second example it is useful to consider two cases depending upon whether $x \geq 0$ or $x \leq 0$.]

14. (Rosen, Example 24, pp. 107 – 108) For each real number **x,** let $\mathrm{int}(x)$ be the greatest integer that is less than or equal to **x.** Prove that

$$\mathrm{int}(2x) = \mathrm{int}(x) + \mathrm{int}(x + \tfrac{1}{2}).$$

15. (Rosen, Exercise 67, p. 111) Prove or disprove the following statements:

- **(1)** For all **x** and **y,** $\mathrm{int}(x + y) = \mathrm{int}(x) + \mathrm{int}(y)$
- **(2)** For all **x** and **y,** $\mathrm{int}(x) + \mathrm{int}(y) + \mathrm{int}(x + y) = \mathrm{int}(2x) + \mathrm{int}(2y)$

16. (Rosen, Exercise 68, p. 111) Prove that

$$\mathrm{int}(3x) = \mathrm{int}(x) + \mathrm{int}(x + \tfrac{1}{3}) + \mathrm{int}(x + \tfrac{2}{3}).$$

17. (Taken from Rosen, Exercise 20, p. 109) Let **f** be a function on the real numbers which is positive valued and let $\mathbf{g} = 1/\mathbf{f}$. Explain why **f** is strictly increasing if and only if **g** is strictly decreasing**.**

18. Prove the **Complement to Proposition IV.4.5** that was stated without proof in the notes**:** *Suppose we have a function* $\mathbf{f} : \mathbf{A} \rightarrow \mathbf{B}$ *and two factorizations of* **f** *as* $\mathbf{j_0} \circ \mathbf{q_0}$ *and* $\mathbf{j_1} \circ \mathbf{q_1}$ *where the maps* $\mathbf{q_t}$ *are surjective and the maps* $\mathbf{j_t}$ *are injective for* $\mathbf{t} = 0, 1.$ *Denote the codomain of* $\mathbf{q_t}$ *(equivalently, the domain of* $\mathbf{j_t}$*) by* $\mathbf{C_t}$. *Then there is a unique bijection* $\mathbf{H}: \mathbf{C_0} \rightarrow \mathbf{C_1}$ *such that* $\mathbf{H q_0} = \mathbf{q_1}$ *and* $\mathbf{j_1 H} = \mathbf{j_0}$. — **[*<u>Hint</u>*:** The mapping **H** should be defined so that if $\mathbf{y} = \mathbf{q_0(x)}$, then $\mathbf{H(y)} = \mathbf{q_1(x)}$. One major step is to show this is well – defined**; *i.e.*,** if $\mathbf{q_0(x)} = \mathbf{q_0(w)}$, then $\mathbf{q_1(x)} = \mathbf{q_1(w)}$. This is one place in the proof where the factorization assumptions play an important role. The next steps are to show that **H** is injective and surjective. Finally, it is necessary to prove the uniqueness of **H.]**

IV.5 : Constructions involving functions

(Halmos, § 8**;** Lipschutz, § 5.7**)**

<u>Problems for study.</u>

Lipschutz : 5.19, 5.49.

<u>Exercises to work.</u>

1. Given two equivalence relations $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ on a set **X,** let $\mathbf{G_1}$ and $\mathbf{G_2}$ be the partitions of **X** that they determine. The ***cross partition*** $\mathbf{G_{1,2}}$ is the partition whose equivalence classes have the form $\mathbf{C} \cap \mathbf{D}$, where **C** is an equivalence class of $\mathbf{G_1}$ and **D** is an equivalence class of $\mathbf{G_2}$. Let $\mathbf{p_1} : \mathbf{X} \rightarrow \mathbf{X}/\boldsymbol{R_1}$ and $\mathbf{p_2} : \mathbf{X} \rightarrow \mathbf{X}/\boldsymbol{R_2}$ be the equivalence class projections, and let $\mathbf{q} : \mathbf{X} \rightarrow (\mathbf{X}/\boldsymbol{R_1}) \times (\mathbf{X}/\boldsymbol{R_2})$ be the map such that the coordinates of **q(x)** are $\mathbf{p_1(x)}$ and $\mathbf{p_2(x)}$ respectively. Prove that the sets in the cross partition are the inverse images of points under the mapping **q.**

2. Prove the exponential laws stated in **Theorem IV.5.5** of the notes**:** *<u>If</u>* **A, B** *<u>and</u>* **C** *<u>are sets, then there is a</u>* $1 - 1$ *<u>correspondence between</u>* $\mathbf{(B \times C)^A}$ *<u>and</u>* $\mathbf{B^A \times C^A}$, *<u>and there is also a</u>* $1 - 1$ *<u>correspondence between</u>* $\mathbf{(C^B)^A}$ *<u>and</u>* $\mathbf{C^{B \times A}}$. — **[*<u>Hints</u>*:** For the first part, let **p** and **q** be the projections from $\mathbf{B \times C}$ to **B** and **C** respectively, then define a map from $\mathbf{(B \times C)^A}$ and $\mathbf{B^A \times C^A}$ sending $\mathbf{f : A \rightarrow B \times C}$ to the ordered pair **(p f, q f),** and show this map is a bijection. For the second part, define mappings

$$\Phi : \mathbf{C^{B\times A}} \rightarrow \mathbf{(C^B)^A}$$

and Ψ in the opposite direction as follows: Given $\mathbf{f : B \times A \rightarrow C}$, let $\Gamma \subset \mathbf{(B \times A) \times C}$ be its graph; for each $\mathbf{a} \in \mathbf{A}$, explain why there is a unique function $\mathbf{g_a : B \rightarrow C}$ whose graph is equal to $q_{B,C}[\Gamma \cap (B \times \{a\}) \times C]$, where $q_{B,C}$ is the projection from the set $(B \times A) \times C$ onto $B \times C$. Define $\Phi(f) = g$, where the value of g at $a \in A$ is the function g_a. Conversely, given $g : A \to C^B$, for each $a \in A$ let $\Gamma(a) \subset (B \times A) \times C$ be the graph of $g(a)$, and show that the set Γ, consisting of all

$$((b, a), c) \in (B \times A) \times C$$

such that $(b, c) \in \Gamma(a)$, is the graph of a function $f: B \times A \to C$, and set $\Psi(g)$ equal to f. Complete the proof by checking that $\Psi\Phi(f) = f$ and $\Phi\Psi(g) = g$ for all f and g.**]**

IV.6 : Order types

(Halmos, § 18**;** Lipschutz, §§ 7.7 – 7.10**)**

Problems for study.

Lipschutz : 7.68, 7.73 – 7.74.

Exercises to work.

1. Show that the subset $[0, 1) \cup [2, 3)$ of the real line has the same order type as the half – open interval $[0,2)$.

2. Show that the subset $[0, 1] \cup [2, 3]$ of the real line does not have the same order type as the half – open interval $[0, 2]$.

3. Let **X** be the power set of an infinite set, and let **Y** be the set of real polynomials with the partial ordering discussed previously, so that both **X** and **Y** are infinite partially ordered sets that are not linearly ordered. Prove that **Y** has the self – density property but **X** does not.

4. Let **D(n)** denote the partially ordered set of positive integers **d** which divide **n,** and take the divisibility relation $a|b$ to be the partial ordering. Prove that **D(28)** and **D(45)** are order – isomorphic, but the sets **D(8)** and **D(15)** are not even though they have the same numbers of elements.

5. Let $\mathbf{N}$ be the nonnegative integers with the usual ordering, and take the lexicographic ordering on $\mathbf{N} \times \mathbf{N}$. Prove that the linearly ordered sets $\mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$ have different order types. **[*Hint:*** For each $x \in \mathbf{N}$, the set of all y such that $y < x$ is finite**.]**

Exercises for Unit V (The basic number systems of mathematics**)**

V.1 : The natural numbers and the integers

(Halmos, §§ 11 – 13**;** Lipschutz, §§ 2.1, 2.7 – 2.9**)**

Problems for study.

Lipschutz : 2.88, 2.92(*c*)

Exercises to work.

1. Suppose we are given a quadratic equation $x^2 + bx + c = 0$ where **b** and **c** are integers, and suppose that **r** is a rational root of this equation. Prove that **r** is an integer. **[*Hint*:** Write the quadratic polynomial as $(x - r)(x - s)$ and explain why $r + s$ and rs must be integers. Why does this imply that **s** is also rational? Next, using the quadratic formula show that $(r - s)^2 = b^2 - 4c$ and hence the right hand side is a is a perfect square, say d^2. Next, write $r = p/q$ where **p** and **q** are relatively prime, and apply the quadratic formula to show that the absolute value of the denominator $|q|$ is at most **2.** Why do this and the integrality of **rs** imply that **b** and **d** must be both even or both odd? Now use the quadratic formula once more to show that the root **r** must be an integer**;** ***i.e.*,** we must have $|q| = 1$.**]**

2. Explain how one can prove irrationality of the square root of **2** from a special case of the preceding result.

3. Let **n** be a positive integer. Explain why the set **A** of all integers greater than or equal to $-n$ is well – ordered. **[*Hint*:** If **B** is a nonempty subset of **A,** consider the set **C** of all integers of the form $n + b$ where $b \in B$.**]**

IV.2 : Finite induction and recursion

(Halmos, §§ 11 – 13**;** Lipschutz, §§ 1.11, 4.6, 11.1 – 11.7**)**

Problems for study.

Lipschutz : 1.36, 1.74 – 1.76

Exercises to work.

1. Prove by induction that for each nonnegative integer **k,** the integer $k^2 + 5k$ is even**.**

2. Prove the summation formula $1^3 + \dots n^3 = (1 + \dots + n)^2 = n^2 (n+1)^2 / 4$.

3. The number **n!** (**n** *factorial*), which is the number of permutations of the first **n** positive integers, is defined recursively by $0! = 1$ and $n! = n \cdot (n-1)!$ for all $n > 0$. Prove that $n! \leq n^n$ for all $n > 0$ and strict inequality holds if $n > 1$.

4. The so – called sequence of ***Fibonacci numbers*** is given recursively by the formulas $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$. Find a function **H** as in the recursive definition theorem which can be used to define the sequence of Fibonacci numbers. **[*Hint*:** The cases $n = 1$ and $n > 1$ must be handled separately.**]**

> ***Comment:*** Fibonacci (*c.* 1170 – 1250**),** whose real name was **Leonardo of Pisa,** made many extremely significant contributions to mathematics, but it is ironic that he is often best known today for mentioning a sequence that had been previously introduced by others and which really plays an extremely minor part in his work as a whole. Further information on these historical issues is given at the following online site:
>
> http://math.ucr.edu/~res/math153/history07.pdf

5. An ***amortized loan*** is paid off in equal periodic payments of **P** units. Assume that the periodic interest rate over the time period is $(100 \cdot r)\%$. If **L** is the loan amount and x_n is the balance remaining after the n^{th} payment, give a recursive formula for x_n.

> ***Comment:*** Usually one is given **L** and **r,** and the objective is to find a value of **P** such that the balance is equal to zero after the M^{th} payment for some fixed **M** (which is normally a multiple of 12**).** In order to do this, one needs to derive a closed formula for x_n and solve for **P** in terms of **L, r** and **M.**

6. Let (N, σ) be a system satisfying the Peano axioms with zero element 0_N. Prove that 0_N is the only element that is not in the image of σ. **[*Hint*:** Apply the third Peano axiom fo the set $A = \{0_N\} \cup \sigma(N)$.**]**

IV.3 : Finite sets

(Halmos, §§ 11 – 13**;** Lipschutz, §§ 1.8, 3.2**)**

Problems for study.

Lipschutz : 1.25, 1.26

Exercises to work.

1. Suppose that we are given two finite sets **A** and **B** and a subset **C** of $\mathbf{A} \times \mathbf{B}$ such that the following hold:

- **(1)** Each element of **A** is the first coordinate for some element of **C.**
- **(2)** For each $\mathbf{a} \in \mathbf{A}$, the number of elements in $\mathbf{C} \cap [\{\mathbf{a}\} \times \mathbf{B}]$ is equal to a fixed constant **k.**

Prove that the number of elements in **C** is equal to $\mathbf{k} \cdot |\mathbf{A}|$, where $|\mathbf{A}|$ denotes the number of elements in **A.** **[*Hint*:** Use induction on the number of elements in **A.]**

2. Use the preceding exercise to compute the number of ordered pairs **(x, y)** where **x** and **y** are integers between **1** and **10** such that one is even and the other is odd.

3. Determine the number of Boolean subalgebras of **P(X)** if **X** is the set **{1, 2, 3, 4}.** How many have exactly **two** atomic subsets?

IV.4 : The real numbers

(Lipschutz, §§ 2.2 – 2.6, 7.7**)**

Problems for study.

Lipschutz : 2.14 – 2.15, 2.17, 2.25 – 2.28, 2.61 – 2.62, 2.67, 2.71 – 2.72, 2.73(*d*), 7.22, 7.25 – 7.26, 7.62 – 7.63, 7.66

Exercises to work.

1. Let **A** be a set of real numbers containing exactly two elements. Explain why **A** is bounded, find the least upper bound and greatest lower bound, and prove that your answers are correct.

2. Let **A** and **B** be a subsets of the real numbers with least upper bounds **u** and **v.** Prove that their union has a least upper bound, and express it in terms of **u** and **v.**

3. Let **A** be the set of negative real numbers. Prove that **0** is equal to the least upper bound of **A.** **[*Hint*:** One needs to check that **0** is an upper bound and if $x < 0$ then **0** is not an upper bound**;** ***i.e.*,** there is some $y \in A$ such that $x < y$.**]**

4. Let **A** be a set of real numbers with least upper bound **x.** Show that there is a sequence of elements x_n in **A** whose limit is equal to **x;** note that the terms of a sequence need not be distinct**.** **[*Hint*:** Given a positive integer **n,** why is there an element **y** of **A** such that $y > a - (1/n)$?**]**

IV.5 : Familiar properties of the real numbers

(Lipschutz, §§ 2.2, 4.5)

Problems for study.

Lipschutz : 4.55, 6.17

Exercises to work.

1. Suppose that **a** and **b** are real numbers such that $a < b$. Prove that there are infinitely many rational numbers in the open interval **(a, b).**

2. For an arbitrary base **N,** one has "base **N** decimal – like" expansions analogous to those for base **10.** In particular, if $k > 1$ is a positive integer then such an expansion

$$1/k = (0.x_1 x_2 x_3 \dots)_N$$

Is given recursively by long division formulas as in the case $N = 10$:

$$N = x_1 k + y_1$$

$$N y_1 = x_2 k + y_2$$

$$\dots$$

Suppose that $N = 16$ and we write the basic hexadecimal digits in the standard form:

1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

Find the base **16** decimal – like expansion for **1/k** where $k = 2, \dots, 10$.

3. Let **f** be the discontinuous, strictly increasing function from the closed unit interval **[0, 1]** to itself which is defined at the end of the section using decimal expansions, and suppose that **x** is an arbitrary point in the unit interval. Explain why **x** is rational if and only if **f(x)** is rational.

4. In the middle to late 14th century Robert Swineshead (or Suiseth, also known as ***the Calculator***) and N. Orseme evaluated the following infinite series (sometimes called the ***Swineshead series***):

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$$

Oresme's technique is equivalent to viewing the latter as a double infinite sum of nonnegative terms

$$\sum \mathbf{a}_{i,j}$$

where $i, j \geq 1$. Standard results in the theory of infinite series (first proved rigorously in the 19th century) imply that the sum of such a series does not change if one rearranges the terms or groups the terms in an arbitrary manner. In particular, if one puts the terms into an infinite matrix where the indices represent the row and column numbers as usual, we can find the value of the sum by first adding along the rows and then along the columns, and similarly we can find the value by first adding along the columns and then along the rows:

$$\sum_{i,j\geq 1} \mathbf{a}_{i,j} = \sum_{i\geq 1}\sum_{j\geq 1} \mathbf{a}_{i,j} = \sum_{j\geq 1}\sum_{i\geq 1} \mathbf{a}_{i,j}$$

A proof of these formulas appears in the following online document:

http://www.math.umn.edu/~jodeit/course/PwrThms.pdf

If we let $\mathbf{a}_{i,j} = 2^{-(i+j)}$ if $i \leq j$ and 0 otherwise, then if we sum first over j holding i fixed, and then we sum over i, we obtain the series displayed above. On the other hand, if we sum first over i holding j fixed, and then we sum over j, we obtain a precise numerical value for the Swineshead series. What is it? **[*Hint*:** Write out the matrix array $\mathbf{a}_{i,j}$ explicitly.**]**

Exercises for Unit VI (Infinite constructions in set theory**)**

VI.1 : Indexed families and set – theoretic operations

(Halmos, §§ 4, 8 – 9**;** Lipschutz, §§ 5.3 – 5.4**)**

Problems for study.

Lipschutz : 5.3 – 5.6, 5.29 – 5.32, 9.14

Exercises to work.

1. Generalize Exercise **12** from Section **III.1** to unions and intersections of arbitrary indexed families of sets**:** Suppose that we have nonempty indexed families of sets $\{ A_j \mid j \in J \}$ and $\{ C_j \mid j \in J \}$ such that $A_j \subset C_j$ for all **j,** Prove the following relationships:

$$\left(\bigcap_{j \in J} A_j\right) \subset \left(\bigcap_{j \in J} C_j\right)$$

$$\left(\bigcup_{j \in J} A_j\right) \subset \left(\bigcup_{j \in J} C_j\right)$$

2. Generalize DeMorgan's laws to unions and intersections of arbitrary indexed families of sets as follows:. Suppose that **S** is a set and we have a nonempty indexed families of subsets of **S** of the form $\{ A_j \mid j \in J \}$**.** Prove the following identities:

$$S - \bigcup_{j \in J} A_j = \bigcap_{j \in J} (S - A_j)$$

$$S - \bigcap_{j \in J} A_j = \bigcap_{j \in J} (S - A_j)$$

3. (Halmos, p. 35) **(a)** Given that $\{ A_j \mid j \in X \}$ and $\{ B_k \mid k \in Y \}$ are nonempty indexed families of sets, prove the following indexed distributive identities:

$$\left(\bigcup_{j \in J} A_j\right) \cap \left(\bigcup_{k \in K} B_k\right) = \bigcup_{j,\,k} (A_j \cap B_k)$$

$$\left(\bigcap_{j \in J} A_j\right) \cup \left(\bigcap_{k \in K} B_k\right) = \bigcap_{j,\,k} (A_j \cup B_k)$$

(b) Suppose that $\{ I_j \mid j \in J \}$ is an indexed family of sets, and write

$$K = \cup \{ I_j \mid j \in J \}.$$

Suppose we are also given an indexed family of sets $\{ A_k \mid k \in K \}$**.** Prove the following identities, assuming in the second case that each of the indexed families is nonempty**:**

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \left(\cup \{ A_i \mid i \in I_j \} \right)$$

$$\bigcap_{k \in K} A_k = \bigcap_{j \in J} \left(\bigcap \{ A_i \mid i \in I_j \} \right)$$

4. (Halmos, p. 37) **(a)** Let $\{ A_j \mid j \in J \}$ and $\{ B_k \mid k \in K \}$ be indexed families of sets. Prove that

$$\left(\bigcup_{j \in J} A_j\right) \times \left(\bigcup_{k \in K} B_k\right) = \bigcup_{j,k} (A_j \times B_k)$$

(another indexed distributive law) and that a similar formula holds for intersections provided that all the indexing sets are nonempty.

(b) Let $\{ X_j \mid j \in J \}$ be an indexed family of sets. Prove that

$$\left(\bigcap_{j \in J} X_j\right) \subset X_k \subset \left(\bigcup_{j \in J} X_j\right)$$

for all $k \in J$. Furthermore, if **M** and **N** are sets such that $M \subset X_j \subset N$ for all **j,** prove that

$$M \subset \left(\bigcap_{j \in J} X_j\right) \quad \text{and} \quad \left(\bigcup_{j \in J} X_j\right) \subset N.$$

VI.2 : Infinite Cartesian products

(Halmos, § 9**;** Lipschutz, §§ 5.4, 9.2**)**

Problems for study.

Lipschutz : 5.11

Exercises to work.

1. (**"**A product of products is a product.**"**) Let X_j be a family of nonempty sets with indexing set **J,** and let $J = \bigcup \{ J_k \mid k \in K \}$ be a partition of **J.** Construct a bijective map from $\prod_j X_j$ to the set

$$\prod_{k \in K} \left(\prod \{ X_j \mid j \in J_k \} \right).$$

[*Hint*: Use the Universal Mapping Property.**]**

2. Let **J** be a set, and for each $j \in J$ let $f_j : X_j \to Y_j$ be a set – theoretic map. Prove that there is a unique map

$$F = \prod_j f_j : \prod_j X_j \to \prod_j Y_j$$

defined by the conditions

$$p_j^Y \circ F = f_j \circ p_j^X$$

where $\mathbf{p_j}^{\mathbf{X}}$ and $\mathbf{p_j}^{\mathbf{Y}}$ denote the $\mathbf{j}^{\text{th}}$ coordinate projections for $\prod_j \mathbf{X_j}$ and $\prod_j \mathbf{Y_j}$ respectively. Also prove that this map is the identity map if each $\mathbf{f_j}$ is an identity map. Finally, if we are also given sets $\mathbf{Z_j}$ with maps $\mathbf{g_j : Y_j \to Z_j}$, and $\mathbf{G} = \prod_j \mathbf{g_j}$, then show that $\mathbf{G \circ F} = \prod_j (\mathbf{g_j \circ f_j})$.

> ***Notation.*** The map of products $\prod_j \mathbf{f_j}$ constructed in the preceding exercise is frequently called the ***product*** of the maps $\mathbf{f_j}$.

3. Let $\{\mathbf{X_j}\}$ and $\{\mathbf{Y_j}\}$ be indexed families sets with the same indexing set **J,** and assume that for each $\mathbf{j} \in \mathbf{J}$ the mapping $\mathbf{f_j : X_j \to Y_j}$ is a bijection. Prove that the product map $\prod_j \mathbf{f_j} : \prod_j \mathbf{X_j} \to \prod_j \mathbf{Y_j}$ is also a bijection. **[*Hint*:** What happens when one takes the product of the inverse maps?**]**

4. Suppose in the preceding exercise we only know that each mapping $\mathbf{f_j}$ is an injection or each mapping $\mathbf{f_j}$ is a surjection. Is the corresponding statement true for the product map? In each case either prove the answer is yes or find a counterexample.

> ***Coequalizers.*** Here is another fundamental example of a universal mapping property. Given two functions $\mathbf{f, g : A \to B}$, a ***coequalizer*** of **f** and **g** is defined to be a map $\mathbf{p : B \to C}$ such that $\mathbf{p\,f = p\,g}$ which has the following universality property: Given an arbitrary map $\mathbf{q : B \to D}$ such that $\mathbf{q\,f = q\,g}$, then there exists a unique mapping $\mathbf{h : C \to D}$ such that $\mathbf{q = h\,p}$. **—** In geometrical studies, such constructions arise naturally if one tries to build an object out of two simpler pieces by gluing them together in some manner (say along their edges), and there are also numerous other mathematical situations where examples of this concept arise.

5. Prove that every pair of functions $\mathbf{f, g : A \to B}$ has a coequalizer. **[*Hint*:** Consider the equivalence relation generated by requiring that **f(x)** be related to **g(x)** for all **x** in **A.]**

6. In the setting of the previous exercise, suppose that $\mathbf{p : B \to C}$ and $\mathbf{r : B \to E}$ are coequalizers of **f** and **g.** Prove that there is a unique bijection $\mathbf{H : C \to E}$ such that $\mathbf{r = H\,p}$. **[*Hint*:** Imitate the proof of the corresponding result for products.**]**

VI.3 : Transfinite cardinal numbers

(Halmos, §§ 22 – 23**;** Lipschutz, §§ 6.1 – 6.3, 6.5**)**

Problems for study.

Lipschutz : 6.4, 6.12

Exercises to work.

1. (Halmos, p. 92) Prove that the set **F(S)** of finite subsets of a countable set **S** is countable, and it is (countably) infinite if and only if **S** is (countably) infinite.

2. Suppose that ***E*** is an equivalence relation on a countably infinite set **S,** and let **S/*E*** be the associated family of equivalence classes. Explain why **S/*E*** is countable.

VI.4 : Countable and uncountable sets

(Halmos, §§ 23 – 23; Lipschutz, §§ 6.3 – 6.7)

Problems for study.

Lipschutz : 6.2 – 6.3, 6.14, 6.32

Exercises to work.

1. (Halmos, p. 95) Let α, β, γ, δ be cardinal numbers such that $\alpha \leq \beta$ and $\gamma \leq \delta$. Prove that $\alpha + \gamma \leq \beta + \delta$ and $\alpha \cdot \gamma \leq \beta \cdot \delta$.

2. Let $\alpha \neq 0$ be a cardinal number. Prove that $\alpha \cdot 0 = 0$, $\alpha^1 = \alpha$ and $1^\alpha = 1$.

3. Let $\Sigma(\mathbf{R})$ denote the set of all $1 - 1$ correspondences from the real numbers to itself. Prove that the cardinal number of $\Sigma(\mathbf{R})$ is equal to $2^{|\mathbf{R}|}$. **[*Hint*:** Why is $2^{|\mathbf{R}|}$ equal to $|\mathbf{R}|^{|\mathbf{R}|}$**?** Why is $\Sigma(\mathbf{R})$ a subset of $\mathbf{R}^{\mathbf{R}}$ and what conclusion does this yield? Next, for each subset of $\mathbf{R}$ define a $1 - 1$ correspondence from $\mathbf{R}$ to itself as follows: Since we have $|\mathbf{R}| + |\mathbf{R}| = |\mathbf{R}|$, it follows that we can partition $\mathbf{R}$ into two pairwise disjoint subsets **A** and **B** that are each in $1 - 1$ correspondence with $\mathbf{R}$; let **f** and **g** be $1 - 1$ correspondences from $\mathbf{R}$ to **A** and **B** respectively. For $C \subset \mathbf{R}$, define a $1 - 1$ correspondence h_C such that h_C interchanges **f(t)** and **g(t)** for each $t \in C$ and $h_C(x) = x$ otherwise. Why are h_C and h_D unequal if $C \neq D$? Look at the set of all $y \in A$ such that $h_C(y) \neq y$, and use this to conclude that there is a $1 - 1$ mapping from $P(\mathbf{R})$ into $\Sigma(\mathbf{R})$.**]**

4. Prove that the set of countable subsets of the real numbers has the same cardinality as the real numbers themselves.

5. It is known that a continuous function on an interval in the real numbers is completely determined by its values at rational points. What does this imply about the cardinal number of continuous functions on an interval?

6. What is the cardinal number of the set of all partial orderings on **N** (the nonnegative integers)? **[*Hint:*** There is a **1 – 1** correspondence between binary relations and subsets of **N** × **N.** What upper bound does this yield for the set of all partial orderings? For every subset **A** of **N** with more than one element, consider the partial ordering which agrees with the usual one on **A** but is modified so that no elements in the complement **N – A** are comparable to any other elements in **N.** Why do different subsets determine different partial orderings? Think about the collection of isolated elements that are not comparable to anything other than themselves. How many subsets of this type are there in **N?** — ***Note:*** A considerably more difficult version of this exercise is to show that the cardinality of the set of all partial orderings on **N** is equal to the cardinality of the set of all order types of partial orderings on **N.]**

VI.6 : Transfinite induction and recursion

(Halmos, §§ 12 – 13, 17 – 20**;** Lipschutz, §§ 8.1 – 8.9, 8.12 – 8.13**)**

Problems for study.

Lipschutz : 8.21, 8.22

Exercises to work.

1. (Halmos, p. 68) A subset **C** of a partially ordered set **A** is said to be *cofinal* if for each **a** ∈ **A** there is some **c** ∈ **C** such that **c** ≥ **a.** Prove that every linearly ordered set has a cofinal well – ordered subset.

2. (Halmos, p. 69) Prove that a linearly ordered set is well – ordered if and only if the set of strict predecessors of each element is well – ordered.

3. Prove that a well – ordered set is finite if and only it is well – ordered with respect to the opposite ordering. **[*Hint*:** An *infinite* well – ordered set must contain a copy of the first infinite ordinal ω.**]**

Exercises for Unit VII (The Axiom of Choice and related topics**)**

General remark. In all the exercises for this section, the Well – Ordering Principle, the Axiom of Choice, or Zorn's Lemma – or any statement that is shown ***in the course notes*** to follow from these – may be assumed unless explicitly stated otherwise.

VII.1 : Nonconstructive existence statements

(Halmos, §§ 15 – 17**;** Lipschutz, §§ 5.9, 7.6, 9.1 – 9.7**)**

Problems for study.

Lipschutz : 7.16, 9.12

Exercises to work.

1. Use the Axiom of Choice to prove the following statement: If $f : A \to B$ is a function, then there is a function $g : B \to A$ such that $f = f g f$.

2. Let **A** and **B** be sets. Prove that $|A| \leq |B|$ if and only if there is a surjection from **B** to **A. [*Hint* :** One implication direction is in the notes for this section, and the other is in the exercises for Section **IV.4**. **]**

DEFINITION(S). It is possible to define ***transfinite arithmetic operations*** on cardinal numbers. Specifically, if we are given an indexed family of cardinal numbers α_j (with indexing set **J**) and sets X_j such that $|X_j| = \alpha_j$, then the *transfinite product* $\prod_j \alpha_j$ is equal to the cardinality of $\prod_j X_j$. According to Exercise **3** in Section **VI.2,** this cardinal number does not depend upon the choice of the indexed family of sets X_j (assuming these sets satisfy $|X_j| = \alpha_j$ for all **j**).

We would like to define a corresponding transfinite sum of the cardinal numbers.

Given an indexed family of sets Y_k with indexing set **K,** the ***disjoint union*,** sometimes also called the ***set – theoretic sum*,** is defined to be the set

$$\coprod_k Y_k = \{ (y, q) \in (\cup_k Y_k) \times K \mid y \in Y_q \}.$$

3. In the setting above, let **W** be the set defined in the displayed equation, and define W_q to be the set of all points in **W** whose second coordinate is equal to **q.** Prove that the sets $\mathbf{W_q}$ are disjoint, their union is all of $\coprod_k \mathbf{Y_k}$, and for each $\mathbf{q}$ we have $|\mathbf{W_q}| = |\mathbf{Y_q}|$.

4. In the setting above, suppose that we are given a second indexed family of sets $\mathbf{V_k}$ with the same indexing set $\mathbf{K}$, and that for each $\mathbf{k}$ in $\mathbf{K}$ we have a bijection $\mathbf{f_k}$ from $\mathbf{Y_k}$ to $\mathbf{V_k}$. Prove that there is a bijection from $\coprod_k \mathbf{Y_k}$ to $\coprod_k \mathbf{V_k}$.

> ***Consequence and definition.*** By the conclusion of the preceding exercise, if we are given an indexed family of cardinal numbers α_j as above and we set the *transfinite sum* $\Sigma_j\ \alpha_j$ equal to the cardinality of the set $\coprod_j \mathbf{X_j}$, then this cardinal number does not depend upon the choice of indexed family $\mathbf{X_j}$ such that $|\mathbf{X_j}| = \alpha_j$.

Footnote. The set – theoretic *sum* or *disjoint union* construction has numerous formal properties that we shall not discuss in this course. Further information may be found in Section **V**.2 of the online notes

http://math.ucr.edu/~res/math205A/gentopnotes.pdf

and the corresponding exercises in the following online document:

http://math.ucr.edu/~res/math205A/gentopexercises.pdf

VII.2 : Extending partial orderings

(Lipschutz, §§ 7.6)

Problems for study.

Lipschutz : 7.16 – 7.18

Exercises to work.

1. (Taken from Rosen, Exercise 55, p. 530) Find a compatible linear ordering for the partial ordering in Exercise 26 on pp. 528 – 529 of Rosen (see Exercise **6** for Section **IV.2**).

2. (Rosen, Exercise 56, p. 530) For the partial ordering on the subset of positive integers

$$\{1, 2, 3, 6, 8, 12, 24, 36\}$$

determined by divisibility, find a linear ordering containing it.

3. (Taken from Rosen, Exercise 58, p. 530) Suppose that we are given a set of tasks

A, B, C, D, E, F, G, H, K, L, M

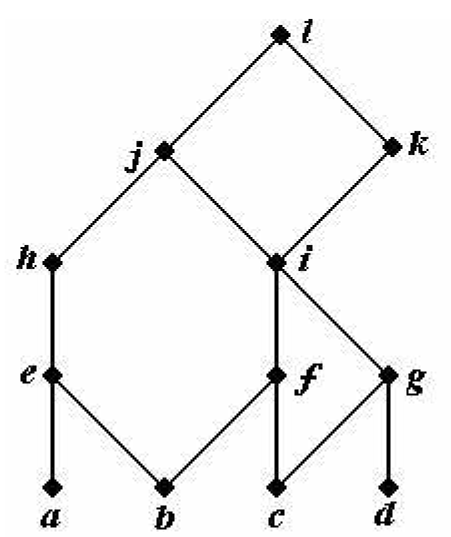
that need to be completed to finish a job, and that they must be scheduled as indicated below:

A	must precede	**B**
B	must precede	**C**
C	must precede	**D**
D	must precede	**E**
E	must precede	**F**
A	must precede	**G**
G	must precede	**H**
G	must precede	**C**
H	must precede	**K**
H	must precede	**D**
K	must precede	**F**
A	must precede	**L**
L	must precede	**M**
M	must precede	**F**

Find a scheduling of the tasks that is compatible with these conditions. **[*Hint :*** View the list as defining a partially ordered set, and draw a Hasse diagram to represent this partially ordered set. Then find a compatible linear ordering for the set**.]**

4. Find a compatible linear ordering for the partially ordered set **P(X),** where **X** = **{1, 2, 3}.**

5. Find a compatible linear ordering for the partially ordered set with the following Hasse diagram:



6. Suppose that ***L*** is a linear ordering on a set **X** which contains more than two elements. Prove that ***L*** contains a partial ordering ***P*** which is ***not*** a linear ordering. **[*Hint :*** Let **A** be a subset of **X** with three elements; take ***P*** so that no elements in **X – A** are comparable to each other and the restriction **P|A** is not a linear ordering.**]**

VII.3 : Equivalence proofs

(Halmos, §§ 15 – 17**;** Lipschutz, §§ 5.9, 7.6, 9.1 – 9.7**)**

Problems for study.

Lipschutz : 7.16, 9.12

Exercises to work.

1. Prove the following result, which is independently due to J. W. Tukey (1915 – 2000) and O. Teichmüller (1913 – 1943), and is generally known as **Tukey's Lemma:** Let **F** be a family of subsets of a fixed set **X,** and assume that it has ***finite character*;** ***i.e.*,** a set **A** lies in **F** if and only if every finite subset of **A** lies in **F.** Then **F** has a maximal element. **[** ***Example*:** The linearly independent subsets of a vector space form a family of finite character. **]**

2. Let **S** be a set, and let $\mathbf{F} \subset \mathbf{P(S)}$ be a collection of pairwise disjoint subsets. Prove that there is a subset **C** of **S** that has ***exactly one element in common with each*** subset **A** in **F.**

VII.4 : Additional consequences

(Halmos, §§ 15 – 17**;** Lipschutz, §§ 5.9, 7.6, 9.1 – 9.7**)**

Problems for study.

Lipschutz : 7.16, 9.12

Exercises to work.

1. (Halmos, p. 95) Let α_j and β_j be indexed families of cardinal numbers with indexing set **J** such that $\alpha_j < \beta_j$ for all $j \in J$. Prove that $\Sigma_j \alpha_j < \Pi_j \beta_j$. **[** ***Hint*:** For each $j \in J$ let X_j and Y_j be sets such that $|X_j| = \alpha_j$ and $|Y_j| = \beta_j$. It will suffice to show that there is no surjection from $\coprod_j X_j$ to $\Pi_j Y_j$. Use a modified Cantor diagonal process argument to show that any map from the first set to the second is not onto. **]**

2. In the setting of the preceding exercise, what conclusion (if any) can be drawn if the inequalities of cardinal numbers are not necessarily strict and all the cardinal numbers in sight are transfinite? Prove your assertion or give examples**. [** ***Remark*:** The hypothesis that all cardinal numbers under consideration are infinite is added to make the proof simpler; it allows one to assume that **|A|** + **1** = **|A|** for all sets **A** that arise in the discussion**.]**

3. (Halmos, p. 96) Suppose that α, β and γ are cardinal numbers and $\alpha \le \beta$**.** Prove that $\alpha^\gamma \le \beta^\gamma$**.** Also prove that if α and β are finite but greater than **1** and γ is infinite, then $\alpha^\gamma = \beta^\gamma$**.**

4. (Halmos, p. 100) If **X** is an infinite set, let $\lambda_0(X)$ be the least ordinal λ such that there is a bijection from λ to **X;** as indicated in the notes, the existence of such an ordinal is a consequence of the Well – Ordering Principle**.** Explain why $\lambda_0(X)$ is a limit ordinal.

5. If we define cardinal numbers to be equal to specific ordinal numbers as in the preceding exercise, which is the first ordinal that is ***not*** equal to a cardinal number?

6. (Halmos, p. 101) If **A** is an infinite set, what is the cardinality of the set of all countable subsets of **A**? **[*Hint*:** There are two cases depending upon whether or not **|A| > |R|.]**

7. Explain why there is a first ordinal Λ_1 such that $|\Lambda_1| > \aleph_0$**,** and prove that every *countable* set of ordinals in Λ_1 has a least upper bound in Λ_1**.** The latter is often called the ***first uncountable ordinal.***

8. If **S** is a set, then a family of subsets **F** of **S** has the ***finite intersection property*** if for every finite subfamily $\{A_1, \dots, A_n\}$ of **F** the intersection $\cap_j A_j$ is nonempty. Prove that if **F** has the finite intersection property, then **F** is contained in a maximal family of subsets which has the finite intersection property.