Mathematics 144, Winter 2022, Review for Examination 1

The examination will consist of four problems, most if not all of which are closely related to the ones given below.

0. Know how to work the following problems from Cunningham:

Exercises 1.1: 1–7

Exercises 1.2: 12–15, 18–21, 27, 29

1. (*a*) Let $\mathbb{N}_+$ denote the positive integers, let $\mathcal{R}$ be the binary relation $x \mathcal{R} y$ if and only if there are odd positive integers a and b such that $xa = yb$. Prove that $\mathcal{R}$ is an equivalence relation, and explain why each equivalence class contains a unique integer of the form 2^m where $m \in \mathbb{N}$.

(*b*) Let $\mathbb{N}_+$ denote the positive integers, let $\mathcal{S}$ be the binary relation $x \mathcal{S} y$ if and only if $y = 2x^m$ where $m \in \mathbb{N}$. Show that $\mathcal{S}$ is not an equivalence relation.

2. Give examples of sets A and B which show that $(A - B) \times (B - A)$ might be empty or nonempty. Drawing pictures will probably be helpful for solving this question.

3. Let X be a set, and let $\mathcal{R}$ be a binary relation which is both an equivalence relation and a partial ordering. Prove that $x \mathcal{R} y$ if and only if $x = y$.

4. Let X be a set, and let $f : X \to X$ and $g : X \to X$ be 1–1 onto functions. Prove that $g^{-1} \circ f$ is also a 1–1 onto function and prove that its inverse is $f^{-1} \circ g$.

5. If $f : \mathbb{R} \to \mathbb{R}$ is the function $f(x) = x|x|$, then f is a 1–1 onto function from $\mathbb{R}$ to itself. Express the inverse function f^{-1} explicitly in terms of x. It is probably useful to split this into two cases depending upon whether $x \geq 0$ or $x \leq 0$.

6. Let X be a partially ordered set with orderinng relation $\leq$, and suppose that a and b are distinct maximal elements of X (we say $m \in X$ is maximal if there is no $u \in X$ such that $m < u$). Prove that the partial ordering is not a linear ordering.

7. Let $\mathbf{P}$ denote the set of polynomials whose coefficients are nonnegative integers, and define a binary relation by $g|f$ (g divides f with no remainder) if and only if $f = gh$ for some $h \in \mathbf{P}$. Prove that this relation is a partial ordering on $\mathbf{P}$. Here is a hint: If h_1 and h_2 are in $\mathbf{P}$, explain why their product also lies in $\mathbf{P}$.