Mathematics 144, Winter 2022, Review for Examination 1

Solutions

Cunningham, Exercises 1.1

1. Let X be a large set containing A and B. Then the hypothesis $a \notin A - B$ translates to $a \notin A \cap (X - B)$, so that $a \in X - (A \cap (X - B)) = (X - A) \cup B$. Since we also have $a \in A$, the only alternative is $a \in B$.∎

2. Taking contrapositives, we conclude that $x \notin B$ implies $x \notin A$. Therefore $x \in C - B$ implies $x \in C - A$.∎

3. We are given that $x \in A$ and $x \notin B$ implies $x \in C$. We want to prove that $y \in A$ and $y \notin C$ implies $y \in B$. Suppose this is false, so that we also have $y \notin B$. Then the first sentence implies that $y \in C$, contradicting our assumption that $y \notin C$. The source of this contradiction was our assumption that $y \notin B$, so this must be false and therefore we must have $y \in B$.∎

4. If $x \in A$ then the hypotheses imply $x \in B$ and $x \in C$, so that $x \in B \cap C$, and therefore $x \in B \cap C$.∎

5. If $x \in A$, then $A \subset B$ implies $x \in B$. Therefore $B \cap C = \emptyset$ implies $x \notin C$, so that $x \in B = C$ and hence $x \in B - C$. It follows that $A \subset B - C$.∎

6. Let $S = A \cup B \cup C$. If $x \in A - (B - C)$ then by definition

$$x \in A \cap (S - (B - C)) \quad = \quad A \cap (S - (B \cap (S - C))) \quad = \quad A \cap ((S - B) \cup C) \quad =$$

$$(A \cap (S - B)) \cup (A \cap C)$$

and since we have $A - B = A \cap (S - B)$ and $A \cup C \subset C$ it follows that $x \in (A - B) \cup C$.∎

7. First of all, A is not empty because the empty set is contained in C for every choice of C. If $x \in A$ but $x \notin C$, then $A - B \subset C$ implies that $x \notin A \cap B$. Since $A = (A - B) \cap (A \cap B)$, this forces the conclusion that $x \in A \cap B$, and hence the latter is nonempty.∎

12. This is one of the two DeMorgan Laws.■

13. This is the other DeMorgan Law.■

14. Suppose that $x \in (A - B) \cap (C - B)$. Then $x \in A - B$ and $x \in C - B$ imply $x \in A$ and $x \in C$, so that $x \in A \cap C$. Since we are also given that $x \notin B$, it follows that $x \in (A \cap C) - B$, so that $(A - B) \cap (C - B) \subset (A \cap C) - B$. Conversely, if $x \in (A \cap C) - B$, then $x \in A$, $x \in C$ and $x \notin B$ imply $x \in A - B$ and $x \in C - B$ implies $x \in (A - B) \cap (C - B)$, so that $(A \cap C) - B \subset (A - B) \cap (C - B)$.■

15. These were done in the course notes.■

18. Suppose that $x \in A$. then $A \subset B$ and $B \cap C = \emptyset$ imply that $x \in B$ but $x \notin C$, so that $x \in B - C$.■

19. We are given that $x \in A - B \subset C$ and $A \not\subset C$. The last condition implies that there is some $x \in A$ such that $x \notin C$. We claim that $x \in B$; if this were false, then we would have $x \in A - B$, and since $A - B \subset C$ it follows that $x \in C$. This contradicts our choice of x such that $x \notin C$. The source of the contradiction is our assumption that $x \notin B$, so we must have $x \in B$. Since we are given $x \in A$ it follows that $x \in A \cap B$ and hence $A \cap B$ is nonempty.■

20. Suppose first that $A \subset B$. If $C \in \mathcal{P}(A)$ then $C \subset A$, and since $A \subset B$ we also have $C \subset B$, so that $C \in \mathcal{P}(B)$.

Conversely, suppose that $\mathcal{P}(A) \subset \mathcal{P}(B)$. If $x \in A$ then $\{x\} \in \mathcal{P}(A) \subset \mathcal{P}(B)$, and since $x \in \mathcal{P}(B)$ we must have $x \in B$ because B is the big union of the family $\mathcal{P}(B)$.■

21. If $E \in \mathcal{P}(A) \cup \mathcal{P}(B)$ then $E \subset A$ or $E \subset B$. In either case we have $E \subset A \cup B$, which means that $E \in \mathcal{P}(A \cup B)$. Therefore $\mathcal{P}(A) \cup \mathcal{P}(B) \subset \mathcal{P}(A \cup B)$.■

27. By definition $\bigcup \mathcal{F}$ is the union of all sets C such that $C \in \mathcal{F}$. We are given that $A \subset C_0$ for some $C_0 \in \mathcal{F}$. If $x \in A$, then $A \subset C_0$ implies $x \in C_0$, and by definition of the big union we know that $x \in C_0$ implies $x \in \bigcup \mathcal{F}$. Therefore we conclude that $x \in A$ implies $x \in \bigcup \mathcal{F}$, which means that $A \subset \bigcup \mathcal{F}$.■

29. If $x \in \bigcup \mathcal{F}$ then $x \in C_0$ for some $C_0 \in \mathcal{F}$. The hypotheses imply that $C_0 \subset A$, and therefore $x \in A$. Since C_0 can be an arbitrary set from $\mathcal{F}$, it follows that $\cup \mathcal{F} \subset A$.■

The remaining exercises in `exam1w22review.pdf`

1. (a) The relation $\mathcal{R}$ is reflexive, for if $x \in \mathbb{N}_+$ and $a = b = 1$ then $xa = ab$. Furthermore, $\mathcal{R}$ is symmetric, for if $xa = yb$ where a and b are odd then we also have $yb = xa$ (a and b are still odd!). Finally, the relation is transitive. If $xa = yb$ where a and b are odd and $yc = zd$ where c and d are odd, then $xac = ybc = ybd$; since the product of two odd integers is odd, it follows that ac and bd are both odd, and therefore $x\mathcal{R}z$.

Each equivalence relation contains a unique power of 2 because every positive integer has a unique factorization $2^p q$, where p is a nonnegative integer and q is odd.■

(b) We need to show that one of the three defining properties of an equivalence relation is false for $\mathcal{S}$. The quickest way is to show the relation is not reflexive. This is true because the equation $x = 2x^m$ (where $m \in \mathbb{N}$) is always false if x is a positive integer. In particular it fails for $x = 1$.■

2. The easiest way to construct examples with an empty intersection is to let $A = B$ be an arbitrary set. Then $(A - B) \times (B - A) = \emptyset \times \emptyset = \emptyset$.

To give an example with a nonempty intersection, let $A = \{1, 2\}$ and $B = \{2, 3\}$. Then $A - B = \{1\}$ and $B - A = \{3\}$, so that $(A - B) \times (B - A) = \{(1, 3)\}$.■

3. Since $\mathcal{R}$ is an equivalence relation, we know that it is symmetric and therefore $x \; \mathcal{R} \; y$ implies $y \; \mathcal{R} \; x$. On the other hand, since $\mathcal{R}$ is a partial ordering, we know that it is antisymmetric we know that $x \; \mathcal{R} \; y$ and $y \; \mathcal{R} \; x$ imply $y = x$. Finally, both conditions on $\mathcal{R}$ imply it is reflexive, so that $x \; \mathcal{R} \; x$ for all $x \in X$.■

4. We shall show that $f^{-1} \circ g$ satisfies the composition conditions to be an inverse function. Here are the derivations:

$$\left(g^{-1} \circ f\right) \circ \left(f^{-1} \circ g\right) = g^{-1} \circ f \circ f^{-1} \circ g =$$

$$g^{-1} \circ 1_X \circ g = g^{-1} \circ g = 1_X$$

$$\left(f^{-1} \circ g\right) \circ \left(g^{-1} \circ f\right) = f^{-1} \circ g \circ g^{-1} \circ g =$$

$$g^{-1} \circ 1_X \circ f = f^{-1} \circ f = 1_X$$

The function $f^{-1} \circ g$ is 1–1 and onto because it is a composite of two such functions — namely, g^{-1} and f — and the composite of two 1–1 onto functions also has these properties.■

5. Follow the suggestion and split the problem into two cases; observe that $f(x) = x|x|$ is nonnegative if $x \geq 0$ and nonpositive if $x \leq 0$.

Suppose that $x \geq 0$. Then $f(x) = x^2$ and the inverse function is given by $\sqrt{x}$. On the other hand, if $x \leq 0$ then $f(x) = -x^2$ and the inverse function is given by $-\sqrt{|x|}$. If we define $\operatorname{sgn}(x)$ to be 1 if $x > 0$, $\operatorname{sgn}(0) = 0$, and $\operatorname{sgn}(x) = -1$ if $x < 0$, we can rewrite this as a unified formula $f^{-1}(x) = \operatorname{sgn}(x)\sqrt{|x|}$.■

6. Suppose that the ordering is a linear ordering. Then either $a < b$ or $b < a$. In either case one of a, b is not a maximal element, so we have a contradiction. The source of the contradiction is our assumption that the ordering is a linear ordering, so this must be false.■

7. We begin by verifying the hint: *If* h_1 *and* h_2 *are in* $\mathbf{P}$, *explain why their product also lies in* $\mathbf{P}$. — Since the polynomials are nonzero, we can write them as $h_1(x) = a_m x^m + k_1(x)$ and $h_2(x) = b_n x^n + k_2(x)$ where $a_m, b_n > 0$ and k_1 and k_2 are polymonials of lower degree. This implies that

$$h_1(x)h_2(x) \quad = \quad a_m b_n x^{m+n} \; + \; \text{LOWER DEGREE TERMS}$$

and the terms of lower degree all have nonnegative coefficients since $a_m, b_n > 0$ and the coefficients of k_1 and k_2 are all nonnegative. Therefore the product belongs to $\mathbf{P}$ as claimed.

The binary relation defined in the problem is reflexive because $f = f \cdot 1$ for all f. Furtherore, it is transitive, for if $h|g$ and $g|f$ then $g = hp$ and $f = gq$ for some $p, q \in \mathbf{P}$ and therefore $f = (qp)h$; by the preceding paragraph we know that the product qp belongs to $\mathbf{P}$. Finally, w se must verify that the relation is antisymmetric. Suppose that $g|f$ and $f|g$, with $f = gp_1$ and $g = fp_2$. Then $f = fp_1p_2$ if we can show that $p_1 = p_2 = 1$ then $f = g$ follows immediately. But $f = fp_1p_2$ implies that the degree of p_1p_2 must be zero, so that the two polynomials in the product must be positive constants. Furthermore, the leading term of f is equal to its product with the positive constants p_1 and p_2. The only way this can happen is if $p_1 = p_2 = 1$, and this implies that $f = g$.■