An informal summary of set theory

Sets are generally viewed as collections of objects. In mathematics the objects under consideration are also thought of as sets.

One fundamental concept is the notion of one set x belonging to another one A, which is written as $x \in A$. Often we say that x ***is an element of*** A.

There are two ways of specifying sets. One is to ***list the members***. For example, the set of all odd positive integers less than 10 is given by $\{1, 3, 5, 7, 9\}$. Another way is to ***describe the property or properties defining the set***. For example, the set discussed previously can be described as

$$\{ x \mid x = 2y + 1 \text{ for some integer } y \textit{ and } 0 < x < 10\}.$$

CAUTION. Some care is needed to avoid statements about collections of objects which are "too large." For example, we cannot discuss the "set of all sets" or other self – referencing statements like "All people are liars" (If a person says this, is the statement true or false? Either answer leads to difficulties). More will be said about this later.

A FEW BASIC EXAMPLES OF SETS:

$\mathbb{N}$	*natural numbers = nonnegative integers*
$\mathbb{Z}$	*(signed) integers*
$\mathbb{Q}$	*rational numbers*
$\mathbb{R}$	*real numbers*
$\mathbb{C}$	*complex numbers*

Set theory would not really be useful for mathematics if the standard number systems could not be viewed as sets, but for most purposes we do not need to know anything significant about the objects which might belong to the sets we think of as numbers.

Some basic relationships and terminology:

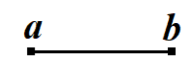
1. Two sets A and B are ***equal***, symbolically $A = B$, if for every $x \in A$ we also have $x \in B$, and for every $x \in B$ we also have $x \in A$.

2. One set A is ***contained in*** (or ***a subset of***) a second set B, written $A \subset B$, if for every $x \in A$ we also have $x \in B$. If in addition $A \neq B$ (the sets are ***not*** equal), then A is said to be ***properly contained in*** (or ***a proper subset of***) B. — In ordinary language, everything which belongs to A also belongs to B, but there is also at least one object which belongs to B but does not belong to A.

3. Two sets A and B are said to be ***disjoint*** if there is no object x such that both $x \in A$ and $x \in B$ are true.

4. There is an ***empty set*** (or ***null set***) $\varnothing$ such that $x \notin \varnothing$ for all x. — We then have $\varnothing \subset A$ for every set A. This is an example of a statement which is ***vacuously true*** because there are no choices of x such that $x \in \varnothing$.

Intervals in the real numbers are fundamentally important examples of sets. They are specified by their upper and lower endpoints, with the convention that there might not be a finite upper or lower endpoint. In each case where there is a finite endpoint, one has intervals where one or both of the endpoints belong to the interval (***closed intervals*** if both do) or one or two endpoints do not belong to the interval (***open intervals*** if neither does). Here are two drawings with comments. A complete description of the formal notation is given on page 3 of Cunningham.



This represents a closed interval $[a, b]$ whose left endpoint is a and whose right endpoint is b. If we remove the two endpoints we obtain an associated open interval (a, b). If we remove just one of the endpoints we obtain a half – open interval.

This represents a closed interval $[a, \infty)$ with left endpoint equal to a and no finite right endpoint. If we remove the endpoint we obtain an associated open interval (a, ∞). The mirror image of the closed interval above (with respect to the vertical axis) is a second closed interval $(-\infty, a]$ which has a right endpoint but no left endpoint.

OPERATIONS AND CONSTRUCTIONS ON SETS: In mathematics one frequently needs to construct new sets from old ones.

Given a set A, one can construct the set of all subsets of A, also called the ***power set*** of A and denoted by $\mathcal{P}(A)$.

Examples. If S is the set $\{1, 2, 3\}$ then there are precisely $8 = 2^3$ subsets in $\mathcal{P}(S)$, and they are all listed below:

$$\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$$

If T is the set $\{1, 2, 3, 4\}$ then there are precisely $16 = 2^4$ subsets in $\mathcal{P}(T)$, and they may be obtained from the list above by (*i*) taking the eight sets in the displayed list, (*i i*) adding the element 4 to each of the eight sets in this list. Clearly one could continue in this fashion to list the subsets of $\{1, 2, 3, 4, 5\}$ and even larger finite sets; in particular, the set of all subsets of $\{1, \dots, n\}$ contains 2^n elements.

Given a set A, the set $\{A\}$, often called ***singleton*** A, is the subset of $\mathcal{P}(A)$ such that A belongs to $\{A\}$ but nothing else belongs to $\{A\}$. As a specal case, one can talk about the set $\{\varnothing\}$ which contains only one element; namely, the empty set. Notice that this set is not the same as the empty set, for $\{\varnothing\}$ is not empty but $\varnothing$ is empty.

The power set construction on a set can be iterated, yielding sets such as $\mathcal{P}(\mathcal{P}(A))$, $\mathcal{P}(\mathcal{P}(\mathcal{P}(A)))$, and so forth.

Example. If S is the set $\{1\}$, then $\mathcal{P}(\mathcal{P}(S))$ consists of the objects $\varnothing$, $\{\varnothing\}$, $\{\{1\}\}$, and $\mathcal{P}(S)$.

BOOLEAN OPERATIONS ON SETS: These are named after the mathematician – philosopher – logician George Boole (1815 – 1864), whose writings described their basic properties.

Let A and **B** be subsets of some set S.

- The ***intersection*** of A and B (often verbalized as A "cap" B) is the set of all elements common to both A and B. It is denoted symbolically by either $A \cap B$ or $\{x \in S \mid x \in A \text{ and } x \in B\}$.
- The ***union*** of two sets A and B (often verbalized as A "cup" B) is the set of elements which are in A or B (and possibly in both). It is denoted symbolically by $\mathbf{A} \cup B$ or $\{x \in S \mid x \in A \text{ or } x \in B\}$.
- The ***relative complement*** of A in B is the set of all elements in B that do not belong to A. It is denoted symbolically by $B - A$ or $B \setminus A$ (in Cunningham) or $\{x \in B \mid x \notin A\}$. Other notations are also used in some contexts.