# Some basic topics in logic

Sections 1.2 – 1.4 of Cunningham discuss some fundamental concepts of mathematical logic from a formal viewpoint. These notes are focused on the concepts which are needed most frequently for mathematical proofs in this and other courses, and the formalities are translated into everyday language.

The concept of a ***mathematical statement*** is fundamental to the discussion which follows. We shall not (and to some extent we cannot) describe this precisely, but the examples should illustrate the concept, and the underlying idea is that "*we know one when we see or hear one*." A mathematical statement may be true or false; an example of a true statement is "for every real number $x$ we have $x0 = 0$," and an example of a false statement is "for every real number $x$ we have $x + x = 0$."

**LOGICALLY EQUIVALENT STATEMENTS:** In mathematical discussions it is important to know about equivalent ways to state the same fact. For example, Problem 2 on page 9 of Cunningham discusses the following equivalence:

***First DeMorgan Law.*** Let **P** and **Q** be mathematical statements. Then the compound statement "(*Either* **P** *is true or* **Q** *is true*) *is false*" is logically equivalent to "*Both* **P** *and* **Q** *are false*."

A formal verification is discussed in Cunningham. Here is an example of how it is used in set theory: $x \notin A \cup B$ is logically equivalent to ($x \notin A$ and $x \notin B$).

There is also a ***Second DeMorgan Law.*** Let **P** and **Q** be mathematical statements. Then the compound statement "(*Both* **P** *and* **Q** *are true*) *is false*" is logically equivalent to "*Either* **P** *is false or* **Q** *is false*."

Here is the corresponding example from set theory: $x \notin A \cap B$ is logically equivalent to ($x \notin A$ or $x \notin B$).

Page 11 of Cunnningham has a fairly long list of other logical equivalences, some of which can be translated into everyday language as indicated:

***First Distributive Law.*** Let **P**, **Q** and **R** be mathematical statements. Then the compound statement "(*Either* **P** *is true or* **Q** *is true*) *and* **R** *is true*" is logically equivalent to "*Either* (*both* **P** *and* **R** *are true*) *or* (*both* **Q** *and* **R** *are true*)."

Here is a corresponding example involving integers: "(*The integer* $p$ *is evenly divisible by* $3$ *or* $5$) *and* ($p$ *is positive*)" is logically equivalent to "*Either* (*the integer* $p$ *is positive and evenly divisible by* $3$) *or* (*the integer* $p$ *is positive and evenly divisible by* $5$)."

***Second Distributive Law.*** Let **P, Q** and **R** be mathematical statements. Then the compound statement "*Either* (*both* **P** *and* **Q** *are true*) *or* (**R** *is true*)" is logically equivalent to "*Both* (*either* **P** *or* **R** *is true*) *and* (**Q** *or* **R** *is true*) *are true*."

Here is a corresponding example involving integers: "(*The integer* $p$ *is evenly divisible by* $5$ *and* $3$) *or* ($p$ *is a perfect square*)" is logically equivalent to "*Both* (*the integer* $p$ *is evenly divisible by* $5$ *or* $p$ *is a perfect square*) *and* (*the integer* $p$ *is evenly divisible by* $3$ *or* $p$ *is a perfect square*) *are true*."

***Double Negation Law.*** Let **P** be a mathematical statement. Then the compound statement "(**P** *is false*) *is false*" is logically equivalent to "**P** *is true*."

***Contrapositive Law.*** Let **P** and **Q** be mathematical statements. Then the compound statement "*If* **P** *is true*, *then* **Q** *is true*" is logically equivalent to "*If* **Q** *is false*, *then* **P** *is false*."

Here is a corresponding example involving integers: "*If the integer* $p$ *is a perfect square*, *then the integer* $p$ *is nonnegative*" is logically equivalent to "*If the integer* $p$ *is negative*, *then the integer* $p$ *is not a perfect square*."

**STATEMENTS WITH VARIABLES:** In ordinary language a statement can be broken down into words or phrases. Similarly, mathematical statements consist of various pieces. It is often helpful to think of some pieces as fixed and others as variable. For example, consider the statement "*Person* $X$ *has blue eyes*." The truth of this statement clearly depends on the identity of "Person $X$." In other words, this is a statement which depends upon the variable $X$, so for each choice of $X$ we obtain a statement **P**($X$) whose validity depends upon the variable "Person $X$." One mathematical example of this sort is "*The number* $x$ *satisfies* $x + 5 = 11$." In this case "*The number* $x$" is the variable, so we know that **P**($x$) is true when $x = 6$ and false otherwise. This is an example of the sorts of statements studied in the ***predicate* (*logical*) *calculus*.**

**STATEMENTS WITH QUANTIFIERS:** Frequently we want to know whether a statement **P**($x$) is true for some or all meaningful values of $x$. This is done by adding the phrase "*for all* [*meaningful choices of*] $x$ ..." or "*for some* [*meaningful choice of*] $x$ ..." (in the second case, this is equivalent to "*there exists some* [*meaningful choice of*] $x$ *such that* ..."). The shorthand symbols $\forall x$ and $\exists x$ for these phrases (respectively) are introduced and studied on page 14 of Cunningham.

Since proofs by contradiction (or reduction ad absurdum) play an important role in mathematics, it is important to understand the precise meaning of a quantified statement. Here are the two basic rules:

The negation of "$\exists x$, **P**($x$) *is true*" is "$\forall x$, **P**($x$) *is false*."

The negation of "$\forall x$, **P**($x$) *is true*" is "$\exists x$, **P**($x$) *is false*."

Frequently mathematical statements have several variable pieces, so one should also understand statements in which two variables are quantified. If the variables have the same quantifiers, then it does not matter which appears first; in symbolic terms, $\exists x \exists y$ is equivalent to $\exists y \exists x$, and $\forall x \forall y$ is equivalent to $\forall y \forall x$. However, if different quantifiers arise then the validity of a statement may depend upon the ordering of the quantifiers. Here is an example involving real numbers:

Let **P**$(x, y)$ be the statement, "$x$ *and* $y$ *are real numbers such that* $x > y$." Then "$\forall y \exists x$, **P**$(x, y)$ *is true*" is true because if we are given $y$ and set $x = y + 1$ then **P**$(x, y)$ is true. On the other hand, "$\exists x \ \forall y$, **P**$(x, y)$ *is true*" is false because there is no real number $x$ which is strictly greater than all real numbers.

Page 19 of Cunningham contains another set of identities called Quantifier Distributive Laws. These deal with situations where **P** can be split into some combination of smaller statements but they are more routine than the preceding material and thus are omitted here. The following two sections (1.4 and 1.5) discuss the formal language for set theory in terms of the logical symbolism introduced up to this point. The material can be skipped without a significant loss of continuity.