Math 144
Fall 2022

SOLUTIONS FOR WEEK 01 EXERCISES

Cunningham, Exercises 1.1

1. We need to show that if $a \in A$ but $a \notin A - B$ implies $a \in B$. By the definitions we know that $A = (A - B) \cup (A \cap B)$. Therefore if a is not in $A - B$ then we must have $a \in A \cap B$. Since $A \cap B$ is contained in B, we must also have $a \in B$.■

3. Let $S = A \cup B \cup C$ so that everything in sight belongs to S. Suppose now that $x \in A - C$, so that $x \in A \cap (S - C)$. By the double negative law and the DeMorgan laws we know that

$$S - C \quad \subset \quad S - (A - B) \ \ S - (A \cap (S - B)) \quad = \quad (S - A) \cup B \ .$$

Since $x \in A - C \subset A$, we know that $x \notin S - A$, so the only remaining alternative is $x \in B$. By definition this implies that $A - C \subset B$.■

6. As in the precedidng exercise, let $S = A \cup B \cup C$. If $x \in A - (B - C)$ then by definition

$$x \in A \cap (S - (B - C)) \quad = \quad A \cap (S - (B \cap (S - C))) \quad = \quad A \cap ((S - B) \cup C) \quad =$$

$$(A \cap (S - B)) \cup (A \cap C)$$

and since we have $A - B = A \cap (S - B)$ and $A \cup C \subset C$ it follows that $x \in (A - B) \cup C$.■

7. First of all, A is not empty because the empty set is contained in C for every choice of C. If $x \in A$ but $x \notin C$, then $A - B \subset C$ implies that $x \notin A \cap B$. Since $A = (A - B) \cap (A \cap B)$, this forces the conclusion that $x \in A\cap$, and hence the latter is nonempty.■

8. $\mathbf{P}(2)$ is true because $2 > \frac{1}{2}$.■
$\mathbf{P}(-2)$ is false because $-2 < -\frac{1}{2}$.■
$\mathbf{P}(\frac{1}{2})$ is false because $1/\frac{1}{2} = 2$ and $2 > \frac{1}{2}$.■
$\mathbf{P}(-\frac{1}{2})$ is true because $1/-\frac{1}{2} = -2$ and $-2 < -\frac{1}{2}$.■

Cunningham, Exercises 1.2

8. By definition $\mathbf{P} \to \mathbf{Q}$ is the statement, "Either (not $\mathbf{P}$) or $\mathbf{Q}$." To streamline the notation we shall denote the negation of $\mathbf{P}$ by $\mathbf{P}^*$. Then $(\mathbf{P} \to \mathbf{Q}) \to \mathbf{R}$ may be rewritten as follows:

$$(\mathbf{P} \to \mathbf{Q}) \to \mathbf{R} \quad \Leftrightarrow \quad (\mathbf{P}^* \text{ or } \mathbf{Q})^* \text{ or } \mathbf{R} \quad \Leftrightarrow \quad (\mathbf{P} \text{ and } \mathbf{Q}^*) \text{ or } \mathbf{R} \quad \Leftrightarrow$$
$$(\mathbf{P} \text{ or } \mathbf{R}) \text{ and } (\mathbf{Q}^* \text{or } \mathbf{R})$$

Similarly, $\mathbf{P} \to (\mathbf{Q} \to \mathbf{R})$ may be rewritten as follows:

$$\mathbf{P}^* \text{ or } (\mathbf{Q} \to \mathbf{R}) \quad \Leftrightarrow \quad \mathbf{P}^* \text{ or } (\mathbf{Q}^* \text{or } \mathbf{R})$$

Therefore the only way that $\mathbf{P} \to (\mathbf{Q} \to \mathbf{R})$ can be false is if both $\mathbf{P}$ and $\mathbf{Q}$ are true but $\mathbf{R}$ is false.

On the other hand, consider when $(\mathbf{P} \to \mathbf{Q}) \to \mathbf{R}$ is false. This happens if either $(\mathbf{P} \text{ or } \mathbf{R})$ or $(\mathbf{Q}^* \text{or } \mathbf{R})$ is false. Therefore $(\mathbf{P} \to \mathbf{Q}) \to \mathbf{R}$ will be false if $\mathbf{R}$ is false, independent of whether $\mathbf{P}$ is true or false and likewise for $\mathbf{Q}$. Consequently there are true/false values for $(\mathbf{P}, \mathbf{Q}, \mathbf{R})$ such that the associated true/false values for

$$(\mathbf{P} \to \mathbf{Q}) \to \mathbf{R} \quad \text{and} \quad \mathbf{P} \to (\mathbf{Q} \to \mathbf{R})$$

are different, which means that the two expressions are not logically equivalent.■

Cunningham, Exercises 1.5

8. The second and third sets are nonempty, but the first one is empty, so the first is not equal to the others. The second set has exactly one element and the third set has exactly two elements, so we also know that the second and third sets must be unequal.■

The remaining exercises in `exercises01.pdf`

3. We claim that $X \subset Y \subset Z$ implies that $X \subset Z$; this is true because $a \in X$ implies $a \in Y$, which in turn implies $a \in Z$. It follows that $A \subset C$, and since $C = A$ it follows that $A = C$. Similarly, $B \subset C \subset A$ implies $B \subset A$, and since $A \subset B$ it follows that $A = B$.■

4. Suppose first that A is properly contained in B, so that there is some $x \in B - A$. Since $B \subset C$ it follows that $b \in C$, so that $b \in C$ but $\notin A$.

Suppose now that B is properly contained in C, so that there is some $x \in C - B$. Then we also have $c \notin A$, for $c \in A$ would implie $c \in B$ and by construction this is not true.■