Assumptions and identities in set theory

We have already noted that some restrictions are needed on the admissible collections of objects in set theory which we call sets; for example, the universal collection $\mathcal{U}$ of all sets is not admissible because we run into difficulties when we consider whether it is an element of itself. Section 2.1 of Cunningham may be viewed informally as a summary of conditions under which a construction on a set will always yield another set.

Roughly speaking, an inadmissible collection is one which is "too large" to be a set, and this is implicit in the axioms for set theory in Section 1.6 of Cunningham. Here is an equivalent restatement of these axioms:

At least one set exists. If this is not true, then nothing else matters.

Subsets. A subcollection of a set is also a set.

Empty set. There is a set which has no elements.

Specification principle. If A is a set and $\mathbf{P}(x)$ is a mathematical statement with variables that is meaningful for every $x \in A$, then

$$\{\, x \in A \mid \mathbf{P}(x) \text{ is true} \}$$

is also a set.

Pairing property. If u and v are sets the there is a set $\{u, v\}$ whose only members are u and v. The case $u = v$ is included, and in this case the set is denoted by $\{u\}$ (or equivalently by $\{v\}$).

Big union property. If $\mathcal{F}$ is a family (= set) of sets, then there is a set $\mathcal{S}(\mathcal{F})$ such that $x \in \mathcal{S}(\mathcal{F})$ if and only if x belongs to some $A \in \mathcal{F}$. Generally we use notation like either $\cup \{A \mid A \in \mathcal{F}\}$ or $\cup_{A \in \mathcal{F}} A$ to specify $\mathcal{S}(\mathcal{F})$.

As noted in Cunningham, the big union property implies the existence of sets such as $\{1, 2, 3\}$. For this example we merely need to take $\mathcal{F}$ to be the family given by $\{1, 2\}$ and $\{3\}$. Continuing in this manner, to obtain $\{1, 2, 3, 4\}$ we take $\mathcal{F}$ to be the family given by $\{1, 2, 3\}$ and $\{4\}$. Clearly this should be repeatable ***ad infinitum***, but we shall not try to do so here.

The usual two set union $A \cup B$ is a special case where $\mathcal{F}$ is given by $\{A\}$ and $\{B\}$.

There is a similar concept of **big intersection** for a **nonempty** family $\mathcal{F}$ of sets:

$$\cap \{ x \in \mathcal{S}(\mathcal{F}) \mid x \in A \text{ for } \underline{\textbf{all}} \ A \in \mathcal{F} \}$$

In this case we must assume that $\mathcal{F}$ is nonempty in order to avoid some logical anomalies described near the bottom of page 33 (and contued on the next page) in Cunningham. A big intersection will automatically be a set because it is a subcollection of each $A \in \mathcal{F}$. We shall also denote this set by $\cap_{A \in \mathcal{F}} A$ or $\cap \{ A \mid A \in \mathcal{F} \}$.

As in the discussion of unions, the usual two set intersection $A \cap B$ is a special case where $\mathcal{F}$ is given by $\{A\}$ and $\{B\}$ if $A \neq B$ and by just $\{A\}$ if $A = B$.

Theorem 2.1.9 describes the relationship of big unions and big intersections if one has two families $\mathcal{F}$ and $\mathcal{G}$ such that one is a subfamily of the other. Read that proof and try to understand it well enough so you could explain the idea correctly (maybe informally) to another student.

Algebraic manipulations in set theory

Section 2.2 of Cunningham considers standard algebraic identities involving unions, intersections and complements. Many reflect the logical equivalences from the previous lecture.

Union and intersection identities. *Let A, B and C be subsets of some fixed universal set $\mathcal{U}$. Then the union and intersection defined as above satisfy the following* ***Boolean algebra*** *identities:*

(**Idempotent Law for unions.**) $A \cup A = A$.

(**Idempotent Law for intersections.**) $A \cap A = A$.

(**Commutative Law for unions.**) $A \cup B = B \cup A$.

(**Commutative Law for intersections.**) $A \cap B = B \cap A$.

(**Associative Law for unions.**) $A \cup (B \cup C) = (A \cup B) \cup C$.

(**Associative Law for intersections.**) $A \cap (B \cap C) = (A \cap B) \cap C$.

(**Distributive Law 1.**) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(**Distributive Law 2.**) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

(**Zero Law.**) $A \cup \varnothing = A$.

(**Unit Law.**) $A \cap \mathcal{U} = A$.

The second group of set – theoretic relations also involves complementation. Frequently the complement of a set A is denoted by something like A', A^*, or A^{c}, particularly if we are discussing relative complements with respect to some fixed semi – universal set $\mathcal{U}$.

Complementation identities. *Let A and B be subsets of some fixed universal set $\mathcal{U}$. Then the union, intersection and relative complement satisfy the following identities:*

(**Double negative Law.**) $(A')' = A$.

(**Complementation Law 1.**) $A \cup A' = \mathcal{U}$.

(**Complementation Law 2.**) $A \cap A' = \emptyset$.

(**De Morgan's Law 1.**) $(A \cup B)' = A' \cap B'$.

(**De Morgan's Law 2.**) $(A \cap B)' = A' \cup B'$.

Verifying these identities. Some of these identities can be checked very simply. For example, the idempotent laws merely reflect the logical tautologies

$$x \in A \Leftrightarrow x \in A \text{ or } x \in A, \quad x \in A \Leftrightarrow x \in A \text{ and } x \in A$$

while the commutative laws follow because both $A \cup B$ and $B \cup A$ are the big union for the family $\{A, B\}$ and also both $A \cap B$ and $B \cap A$ are the big intersection for the family $\{A, B\}$. Similarly, the associative laws follow because both $A \cup (B \cup C)$ and $(A \cup B) \cup C$ are the big union for the family $\{A, B, C\}$ and also both $A \cap (B \cap C)$ and $(A \cap B) \cap C$ are the big intersection for the same family $\{A, B, C\}$. The zero law follows because there is no $x \in \emptyset$, and the unit law follows because A is a subset of $\mathcal{U}$, which implies the intersection identity (since $x \in A$ automatically implies $x \in \mathcal{U}$). To complete the first list of identities, we note that the two distributive laws are direct consequences of the definitions for union and intersection together with the distributive laws for logic in Lecture 02.

Turning to the second list of identities, the double negative law merely reflects the corresponding law for logical statements in Lecture 02, while the complementation laws reflect the facts that the statement $x \in A$ cannot be simultaneously true and false (the intersection identity) and the facts that A is contained in $\mathcal{U}$, so that $A' = \mathcal{U} - A$, plus either $x \in A$ or its negation must be true. To complete the first list of identities, we note that the two DeMorgan laws are direct consequences of the definitions for union and intersection together with the DeMorgan laws for logic in Lecture 02.

Note. Section 2.2 of Cunningham proves much more general versions of the distributive and DeMorgan laws. Specifically, Theorem 2.2.2 proves DeMorgan laws for $A - B$ where A is a set and B is either a big union or a big intersection, and Theorem 2.2.4 proves distributive laws for $A \cup B$ and $A \cap B$, where (as before) A is a set and B is either a big union or a big intersection.