Still more constructions in set theory

At the end of the last lecture we mentioned distributive laws and generalized DeMorgan laws for big unions and big intersections. No proof was described, one reason being that a basic axiom in set theory had not yet been mentioned.

Given a set A, recall that one can construct the set of all subsets of A, also called the ***power set*** of A and denoted by $\mathcal{P}(A)$. The use of the word "set" in this definition is justified by the following assumption:

Power set axiom. If A is a set then the collection $\mathcal{P}(A)$ of all subsets of A is also a set.

This assumption has a simple consequence which we shall need.

Corollary. *If A is a set and $\mathfrak{F}$ is a collection of subsets of A, then $\mathfrak{F}$ is also a set.*

Proof. If $\mathfrak{F}$ is a collection of subsets then $\mathfrak{F}$ is a subcollection of $\mathcal{P}(A)$. Since the latter is a set, the subset property implies that $\mathfrak{F}$ is also a set.

Why is this necessary and helpful? Consider the second part of the DeMorgan Theorem 2.2.2, whose proof was left as an exercise. We want to prove an identity of the form

$$A - (\cup_{B\,\in\,\mathfrak{F}}\, B) \;=\; \cap_{B\,\in\,\mathfrak{F}}\,(A - B)$$

and the previous argument for complements of unions goes through except for one question: ***How do we know that the collection of sets on the right is also a set?*** Fortunately, since each set on the right hand side is a subset of the set A, we can use the preceding corollary to draw this conclusion. In fact, this is Exercise 2.2.1 in Cunningham. The next three exercises 2.2.2 – 2.2.4 are similar results showing that certain other collections of subsets obtained from a family $\mathfrak{F}$ will also be sets, and these guarantee that the expressions on the right hand sides of Theorem 2.2.4 are all sets.

The power set axiom also turns out to be useful in many other contexts. Some will be discussed later in this course.

Read through Exercises 2.2 for several other examples for the algebra of manipulating big unions and intersections. For this course the main goal is to be able to translate the equations into less formal language.

Ordered pairs and Cartesian products

Given sets u and v we can view the set $\{u, v\}$ as describing an ***unordered pair*** of objects. Frequently we wish to think of one element as preceding the other, and this leads to the notion of an ordered pair (u, v). Chapter 3 of Cunningham provides a formal definition of such objects. For most purposes the details of the construction are not needed, and it is enough to use the formal properties, which we summarize here:

Existence of ordered pairs. If u and v are sets the there is an ***ordered pair*** (u, v) such that $(u, v) = (s, t)$ if and only if $u = s$ and $v = t$. Cunningham's notation for an ordered pair is $\langle u, v\rangle$; although there are good reasons for this convention, the notation in these notes is the most widely used in mathematical writings. The set u is said to be the ***first coordinate***, and the set v is said to be ***the second coordinate*** of (u, v).

Perhaps the most familiar examples of ordered pairs appear in analytic (or coordinate) geometry, in which points on a plane correspond to ordered pairs of real numbers. However, there are also other contexts in which ordered pairs are implicit.

Example. One can use ordered pairs to model a standard deck of 52 playing cards. If V is the set of playing card values $\{A, K, Q, J, 10, 9, 8, 7, 6, 5, 4, 3, 2\}$ and S is the set of playing card suits $\{\spadesuit, \heartsuit, \diamondsuit, \clubsuit\}$, then the playing cards can be labeled by ordered pairs (v, s) where $v \in V$ and $s \in S$:

$$\{(A, \spadesuit), (K, \spadesuit), ..., (2, \spadesuit), (A, \heartsuit), ..., (3, \clubsuit), (2, \clubsuit)\}$$

In both coordinate geometry and the card deck model we are considering a collection of all pairs (a, v) where $a \in A$ and $b \in B$ for some sets A and B (in coordinate geometry both sets are the real line). More generally, if we are given sets A and B then we define the ***Cartesian product*** $A \times B$ to be the collection of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

> ***Historical remark.*** Although the name "Cartesian product" is an allusion to the well known work of R. Descartes (1596 – 1650) on introducing algebraic coordinates into geometry, Descartes himself did not explicitly use ordered pairs of numbers (or coordinate axes) to represent points in his writings on coordinate geometry.

He understood the concept of coordinates, but not in the more formal sense of this course.

In order to work effectively with Cartesian products we need the following results:

Cartesian product properties. If A and B are sets, then so is $A \times B$. Furthermore, if A_1 and B_1 are subsets of A and B respectively, then $A_1 \times B_1$ is a subset of $A \times B$.

If we define ordered pairs as in Section 3.1 of Cunningham, then the results of that section yield a proof of these properties. However, for the purposes of this course we can simply assume ordered pairs exist and the Cartesian product is a set.

Big Cartesian products. In analogy with unions and intersections, one might ask if one can define big Cartesian products involving arbitrary families of sets. This can be done, but a concise and general construction requires concepts not yet introduced, so we have to postpone a detailed treatment of this topic until later in the course.

The whiteboard notes for this lecture explain the issues in more detail. There are analogous issues with union and intersection. For example, what is the most convenient way to describe a union of sets A, B, C? We can do this by hand as either (A cup B) cup C or A cup (B cup C), but how do we choose? The big union for the family {A, B, C} doesn't depend upon the way in which we insert parentheses. We would like a similar notion of big Cartesian product for a family of three or more sets. This question will have added importance when we consider infinite sets in more detail.