Math 144
Fall 2022

SOLUTIONS FOR WEEK 02 EXERCISES

Cunningham, Exercises 2.1

1. This was done in the solved examples.■

2. This is done in previous examples.■

3. If $x \in A - C$ then $x \in A$ but $x \notin C$. Since $B \subset C$, we know that $x \notin C$ implies $x \notin B$. But this means that $x \in A - B$, and therefore $A - C \subset A - B$ by the definition of subsets.■

4. This was also done in the solved examples.■

16. Suppose that $x \in (A \cup B) - (A \cap B)$. Then $x \in A$ or $x \in B$ but $x \notin A \cap B$. If $x \in A$ and $x \notin A \cap B$ then $x \in A - (A \cap B)$. However, we also have $x \in A - (A \cap B) = A - B$ because (1) $A \cap B$ is a subset of B, so $x \in A$ and $x \notin B$ implies $x \notin A \cap B$, (2) $x \in A$ and $x \notin A \cap B$ implies $x \notin B$ because $x \in B$ would imply $x \in A \cap B$. Therefore $x \in A - B$. If $x \in B$ we can interchange the roles of A and B in the preceding argument to conclude that $x \in B - A$. Combining these, we have $(A \cup B) - (A \cap B) \subset (A - B) \cup (B - A)$.

Conversely, suppose that $x \in (A - B) \cup (B - A)$. Since $A - B \subset A$ and $B - A \subset B$, it follows that $x \in A \cup B$. Therefore the only remaining issue is to prove that $x \notin A \cap B$. Observe first that $(A - B) \cap (B - A) = \emptyset$, for $x \in A - B$ implies $x \notin B - A$ and $x \in B - A$ implies $x \notin A - B$. If $x \in A - B$ then $A - B = A - (A \cap B)$ implies $x \notin A \cap B$, and likewise if $x \in B - A$ then $B - A = B - (A \cap B)$ implies $x \notin A \cap B$. Therefore $x \notin A \cap B$ in all cases, so that $(A - B) \cup (B - A) \subset (A \cup B) - (A \cap B)$.■

20. Suppose first that $A \subset B$. If $C \in \mathcal{P}(A)$ then $C \subset A$, and since $A \subset B$ we also have $C \subset B$, so that $C \in \mathcal{P}(B)$.

Conversely, suppose that $\mathcal{P}(A) \subset \mathcal{P}(B)$. If $x \in A$ then $\{x\} \in \mathcal{P}(A) \subset \mathcal{P}(B)$, and since $x \in \mathcal{P}(B)$ we must have $x \in B$ because B is the big union of the family $\mathcal{P}(B)$.■

24. This is true because $\emptyset \in \mathcal{P}(A)$ and hence $\emptyset \notin \mathcal{P}(B) - \mathcal{P}(A)$.■

25. We know that $\cup\mathcal{F}$ is the union of all sets A such that $A \in \mathcal{F}$. Since C is a set of the latter type it follows that $x \in C$ implies $x \in \cup\mathcal{F}$.■

26. We know that $\cap\mathcal{F}$ is the intersection of all sets A such that $A \in \mathcal{F}$. Since C is a set of the latter type it follows that $x \in \cap\mathcal{F}$ implies $x \in C$.■

29. If $x \in \cup\mathcal{F}$ then $x \in C_0$ for some $C_0 \in \mathcal{F}$. The hypotheses imply that $C_0 \subset A$, and therefore $x \in A$. Since C_0 can be an arbitrary set from $\mathcal{F}$, it follows that $\cup\mathcal{F} \subset A$.■

30. Since $A \in \mathcal{P}(A)$ we have $A \subset \cup\mathcal{P}(A)$. On the other hand, $C \in \mathcal{P}(A)$ implies $C \subset A$, and therefore $\cup\mathcal{P}(A) \subset A$. Combining these, we see that $\cup\mathcal{P}(A) = A$.■

Cunningham, Exercises 2.2

2. We first claim that the collection $\mathcal{F}'$ of all sets having the form $A \cup C$ for some $C \in \mathcal{F}$ is also a set, which means that the right hand side of the equation in the conclusion will also be a set. Since this new family is contained in $\mathcal{P}\left((\cup(\mathcal{F}) \cup A)\right)$ we know that $\mathcal{F}'$ is indeed a (subcollection of a) set.

Now suppose that $x \in A \cup C$ where $C \in \mathcal{F}$. Then we automatically have $x \in C'$ for some $C' \in \cup\mathcal{F}'$ and therefore $x \in \cup\mathcal{F}'$. Conversely, if $x \in \cup\mathcal{F}'$ then $x \in A \cup C$ for some $C \in F$. Now either $x \in A$ or else $x \in C$. In the second case $x \in \cup\mathcal{F}$, and therefore $x \in A \cup (\cup\mathcal{F})$.

Conversely, if $x \in A \cup (\cup\mathcal{F})$ then either $x \in A$ or else $x \in C$ for some $C \in \mathcal{F}$. In either case $x \in A \cup C$ for some $C \in \mathcal{F}$ and therefore $x \in C'$ for some $C' \in \mathcal{F}'$. Combining the results of this and the preceding paragraph, we see that $\cup\mathcal{F}'$ and $A \cup (\cup\mathcal{F})$ are equal.■

4. For this exercise we need to know that the collection $\mathcal{F}^*$ of all sets having the form $A \cap C$ for some $C \in \mathcal{F}$ is also a set, which means that the right hand side of the equation in the conclusion will also be a set; the collection $\mathcal{F}^*$ will be nonempty because $\mathcal{F}$ is nonempty. This follows by taking the argument in the first paragraph of the preceding solution and replacing $A \cup C$ with $A \cap C$ everywhere.

The solution now proceeds as in the preceding exercise: Suppose that $x \in A \cap C$ where $C \in \mathcal{F}$. Then we automatically have $x \in C'$ for some $C' \in \cup\mathcal{F}^*$ and therefore $x \in \cup\mathcal{F}^*$. Conversely, if $x \in \cup\mathcal{F}^*$ then $x \in A \cap C$ for some $C \in F$. Now both $x \in A$ and $x \in C$. By the second of these, $x \in \cup\mathcal{F}$, and therefore $x \in A \cap (\cup\mathcal{F})$.

Conversely, if $x \in A \cap (\cup\mathcal{F})$ then both $x \in A$ and $x \in C$ for some $C \in \mathcal{F}$. Therefore case $x \in A \cap C$ for some $C \in \mathcal{F}$ and hence $x \in C'$ for some $C' \in \mathcal{F}^*$. Combining the results of this and the preceding paragraph, we see that $x \in \cup\mathcal{F}^*$ and combining this with the preceding paragraph, we conclude that $A \cap (\cup\mathcal{F})$ are equal.■

11. First of all, $\mathcal{F} \cup \mathcal{G}$ is a family of sets because it is the union of the families $\mathcal{F}$ and $\mathcal{G}$.

Suppose that $x \in \cup(\mathcal{F} \cup \mathcal{G})$. Then $x \in C$ where either $C \in \mathcal{F}$ or $C \in \mathcal{G}$. In the first case $x \in \cup\mathcal{F}$ and in the second case $x \in \cup\mathcal{G}$; in either of these cases we have $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$.

Conversely, suppose that $x \in (\cup\mathcal{F}) \cup (\cup\mathcal{G})$. In the first case $x \in C$ for some $C \in \mathcal{F}$ and in the second case $x \in C$ for some $C \in \mathcal{G}$; in either of these cases we have $x \in C$ for some $C \in \mathcal{F} \cup \mathcal{G}$. By the definition of big unions, this means that $x \in \cup(\mathcal{F} \cup \mathcal{G})$.■

The remaining exercises in `exercises02.pdf`

1. $(i) \Rightarrow (ii)$ Since $A \cap B \subset B$, we must have $A = A \cap B \subset B$. $(ii) \Rightarrow (i)$ If $A \subset B$ and $x \in A$, then we also have $x \in B$ and hence $x \in A \cap B$.

$(iii) \Rightarrow (ii)$ Since $A \subset A \cup B$, we must have $A \subset A \cap B = B$. $(ii) \Rightarrow (iii)$ If $A \subset B$ then $x \in A$ implies $x \in B$ so that $x \in A \cup B$ implies $x \in B$ in all cases.■

2. Suppose first that $C \subset A$. Then by distributivity $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$, which is equal to $A \cap (B \cup C)$ by the inclusion hypothesis. Conversely, suppose the modular identity holds for A, B, C. Then we have

$$C \quad \subset (A \cap B) \cup C \quad = \quad A \cap (B \cup C) \quad \subset \quad A$$

as required.■

3. Suppose that $x \in A \cup B$. If $x \in A$ then $c \in C$, and if $x \in B$ then $x \in D$. In either case $x \in C \cup D$, so that $A \cup B \subset C \cup D$.■

Now suppose that $x \in A \cap B$. Since $x \in A$ we have $c \in C$, and since $x \in B$ we have $x \in D$. Therefore $x \in C \cap D$, so that $A \cap B \subset C \cap D$.■

4. (i) Suppose that (x, y) lies in $(A \times B) \cap (C \times D)$. Then we have $x \in A$ and $y \in B$, and we also have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cap C$, while the second and fourth imply $y \in B \cap D$. Therefore $(x, y) \in (A \cap C) \times (B \cap D)$ so that

$$(A \times B) \cap (C \times D) \ \subset \ (A \cap C) \times (B \cap D) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the displayed equation. Then $x \in A \cap C$ and $y \in B \cap D$. Since $x \in A$ and $y \in B$ we have $(x, y) \in A \times B$, and likewise since $x \in C$ and $y \in D$ we have $(x, y) \in C \times D$, so that

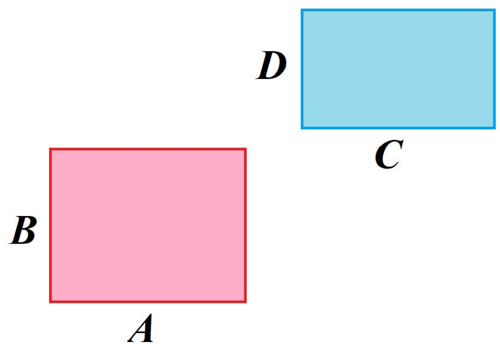
$$(A \times B) \cap (C \times D) \ \supset \ (A \cap C) \times (B \cap D) \ .$$

Therefore the two sets under consideration are equal.■

(*ii*) Suppose that (x, y) lies in $(A \times B) \cup (C \times D)$. Then either we have $x \in A$ and $y \in B$, or else we have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cup C$, while the second and fourth imply $y \in B \cup D$. Therefore (x, y) is a member of $(A \cup C) \times (B \cup D)$ so that

$$(A \times B) \cup (C \times D) \ \subset \ (A \cup C) \times (B \cup D) \ .$$

Supplementary note: To see that the sets are not necessarily equal, consider what happens if $A \cap C = B \cap D = \emptyset$ but all of the four sets A, B, C, D are nonempty. Try drawing a picture in the plane to visualize this.■



(*iii*) Suppose that (x, y) lies in $(X \times Y) - (A \times B)$. Then $x \in X$ and $y \in Y$ but $(x, y) \notin A \times B$. The latter means that the statement

$$x \in A \ \textbf{and} \ y \in B$$

is false, which is logically equivalent to the statement

$$\textbf{either} \ x \notin A \ \textbf{or} \ y \notin B \ .$$

If $x \notin A$, then it follows that $(x, y) \in \big((X - A) \times Y\big)$, while if $y \notin B$ then it follows that $(x, y) \in \big(X \times (Y - B)\big)$. Therefore we have

$$\big(X \times Y\big) - \big(A \times B\big) \ \subset \ \big(X \times (Y - B)\big) \cup \big((X - A) \times Y\big) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the containment relation on the displayed line. Then we have $(x, y) \in X \times Y$ and also

$$\textbf{either} \ x \notin A \ \textbf{or} \ y \notin B \ .$$

The latter is logically equivalent to

$$x \in A \ \textbf{and} \ y \in B$$

and this in turn means that $(x, y) \notin A \times B$ and hence proves the reverse inclusion of sets.■