### SOLVED PROBLEMS FOR WEEK 02

**1.** Let *A* and *B* be sets. Prove the following statements:

- (i) If X is a set such that  $A \subset X$  and  $B \subset X$ , then  $A \cup B \subset X$ .
- (*ii*) If Y is a set such that  $Y \subset A$  and  $Y \subset B$ , then  $Y \subset A \cup B$ .

## SOLUTION.

(i) Suppose that  $x \in A \cup B$ , so either  $x \in A$  or  $x \in B$ . In the first case  $A \subset X$  implies  $x \in X$ , and in the second case  $x \in B$  implies  $x \in X$ , so  $x \in X$  in all cases. By definition, this means  $A \cup B \subset X$ .

(*ii*) Suppose that  $y \in Y$ . Since  $Y \subset A$  this implies  $y \in A$ , and since  $Y \subset B$  this implies  $y \in B$ . Therefore  $y \in A \cap B$ , and hence by definition it follows that  $Y \subset A \cap B$ .

**2.** Let A and B be nonempty sets. Prove that  $A \times B = B \times A$  if and only if A = B.

### SOLUTION.

If A = B then we trivially have  $A \times B = A \times A = B \times A$ . Conversely, suppose we have  $B \times A = A \times B$ . Let  $b \in B$ . Then for each  $x \in A$  we have  $(b, x) \in B \times A = A \times B$ , which means that  $b \in A$ . Thus we have shown that B is contained in A. Similarly, let  $a \in A$ . Then for each  $y \in B$  we have  $(y, a) \in B \times A = A \times B$ , which means that  $a \in B$ . Thus we have shown that B is contained in A. Similarly, let  $a \in A$ .

**3.** Let A, B and C be sets. Prove that  $(A \times B) \cap (A \times C) = A \times (B \cap C)$ .

#### SOLUTION.

Suppose that  $(x, y) \in (A \times B) \cap (A \times C)$ . Then  $(x, y) \in A \times B$  implies  $x \in A$  and  $y \in B$ , and  $(x, y) \in A \times C$  implies  $x \in A$  and  $y \in C$ . Since y belongs to both B and C, we have  $y \in B \cap C$  and hence  $(x, y) \in A \times (B \cap C)$ .

Conversely, if  $(x, y) \in A \times (B \cap C)$  then  $x \in A$  and  $y \in B \cap C$ . The second of these implies that  $(x, y) \in A \times B$  and  $(x, y) \in A \times C$ , and therefore  $(x, y) \in (A \times B) \cap (A \times C)$ .

**4.** Let X be a set and let  $A, B \subset X$ . The symmetric difference  $A \oplus B$  is defined by the formula

$$A \oplus B = (A - B) \cup (B - A)$$

so that  $A \oplus B$  consists of all objects in A or B but not both. Prove that  $\oplus$  satisfies the associative identity  $(A \oplus B) \oplus C = A \oplus (B \oplus C)$  for all  $A, B, C \subset X$ .

# SOLUTION.

Denote the complement X - Y of  $Y \subset X$  by  $Y^*$ , and write  $P \cap Q$  simply as PQ to simplify the algebraic manipulations. Then the symmetric difference can be rewritten in the form  $(AB^*) \cup (BA^*)$ .

It then follows that

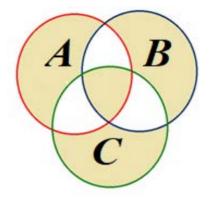
$$(A \oplus B) \oplus C = (AB^* \cup BA^*) C^* \cup C (AB^* \cup BA^*)^* =$$
$$AB^*C^* \cup BA^*C^* \cup C ((A^* \cup B)(B^* \cup A)) = AB^*C^* \cup BA^*C^* \cup C (A^*B^* \cup AB) =$$
$$AB^*C^* \cup A^*BC^* \cup A^*B^*C \cup ABC .$$

Similarly, we have

$$A \oplus (B \oplus C) = A \left( BC^* \cup CB^* \right)^* \cup \left( BC^* \cup CB^* \right) A^* =$$
$$A \left( (B^* \cup C)(C^* \cup B) \right) \cup BC^*A^* \cup CB^*A^* = A(B^*C^* \cup BC) \cup BC^*A^* \cup CB^*A^* =$$
$$AB^*C^* \cup A^*BC^* \cup A^*B^*C \cup ABC .$$

This proves the associativity of  $\oplus$  because the two right hand sides are equal.

Here is a Venn diagram in which the symmetric difference is displayed.



2