Math 144
Winter 2022

SOLVED PROBLEMS FOR WEEK 02

1. Let A and B be sets. Prove the following statements:

(i) If X is a set such that $A \subset X$ and $B \subset X$, then $A \cup B \subset X$.

(ii) If Y is a set such that $Y \subset A$ and $Y \subset B$, then $Y \subset A \cup B$.

SOLUTION.

(i) Suppose that $x \in A \cup B$, so either $x \in A$ or $x \in B$. In the first case $A \subset X$ implies $x \in X$, and in the second case $x \in B$ implies $x \in X$, so $x \in X$ in all cases. By definition, this means $A \cup B \subset X$.■

(ii) Suppose that $y \in Y$. Since $Y \subset A$ this implies $y \in A$, and since $Y \subset B$ this implies $y \in B$. Therefore $y \in A \cap B$, and hence by definition it follows that $Y \subset A \cap B$.■

2. Let A and B be nonempty sets. Prove that $A \times B = B \times A$ if and only if $A = B$.

SOLUTION.

If $A = B$ then we trivially have $A \times B = A \times A = B \times A$. Conversely, suppose we have $B \times A = A \times B$. Let $b \in B$. Then for each $x \in A$ we have $(b, x) \in B \times A = A \times B$, which means that $b \in A$. Thus we have shown that B is contained in A. Similarly, let $a \in A$. Then for each $y \in B$ we have $(y, a) \in B \times A = A \times B$, which means that $a \in B$. Thus we have shown that A is contained in B. Combining these, we conclude that $A = B$.■

3. Let A, B and C be sets. Prove that $(A \times B) \cap (A \times C) = A \times (B \cap C)$.

SOLUTION.

Suppose that $(x, y) \in (A \times B) \cap (A \times C)$. Then $(x, y) \in A \times B$ implies $x \in A$ and $y \in B$, and $(x, y) \in A \times C$ implies $x \in A$ and $y \in C$. Since y belongs to both B and C, we have $y \in B \cap C$ and hence $(x, y) \in A \times (B \cap C)$.

Conversely, if $(x, y) \in A \times (B \cap C)$ then $x \in A$ and $y \in B \cap C$. The second of these implies that $(x, y) \in A \times B$ and $(x, y) \in A \times C$, and therefore $(x, y) \in (A \times B) \cap (A \times C)$.■

4. Let X be a set and let A, $B \subset X$. The *symmetric difference* $A \oplus B$ is defined by the formula

$$A \oplus B \quad = \quad (A - B) \ \cup \ (B - A)$$

so that $A \oplus B$ consists of all objects in A or B but not both. Prove that $\oplus$ satisfies the associative identity $(A \oplus B) \oplus C \ = \ A \oplus (B \oplus C)$ for all $A, B, C \subset X$.

SOLUTION.

Denote the complement $X - Y$ of $Y \subset X$ by Y^*, and write $P \cap Q$ simply as PQ to simplify the algebraic manipulations. Then the symmetric difference can be rewritten in the form $(AB^*) \cup (BA^*)$.

It then follows that

$$(A \oplus B) \oplus C = (AB^* \cup BA^*)\, C^* \,\cup\, C\,(AB^* \cup BA^*)^* =$$

$$AB^*C^* \cup BA^*C^* \cup C\,((A^* \cup B)(B^* \cup A)) = AB^*C^* \cup BA^*C^* \cup C\,(A^*B^* \cup AB) =$$

$$AB^*C^* \cup A^*BC^* \cup A^*B^*C \cup ABC\ .$$

Similarly, we have

$$A \oplus (B \oplus C) = A\,(BC^* \cup CB^*)^* \cup (BC^* \cup CB^*)\,A^* =$$

$$A\,((B^* \cup C)(C^* \cup B)) \cup BC^*A^* \cup CB^*A^* = A(B^*C^* \cup BC) \cup BC^*A^* \cup CB^*A^* =$$

$$AB^*C^* \cup A^*BC^* \cup A^*B^*C \cup ABC\ .$$

This proves the associativity of $\oplus$ because the two right hand sides are equal.∎

Here is a Venn diagram in which the symmetric difference is displayed.