Math 144
Winter 2022

EXERCISES FOR WEEK 03

1. Determine whether each of the following binary relations on the set $\{1, 2, 3, 4\}$ is reflexive, symmetric or transitive, and give reasons for your responses.

(a) $\{ (2,2),\ (2,3),\ (2,4),\ (3,2),\ (3,3),\ (3,4) \}$

(b) $\{ (1,1),\ (2,2),\ (2,1),\ (1,2),\ (3,3),\ (4,4) \}$

(c) $\{ (2,4),\ (4,2) \}$

(d) $\{ (1,2),\ (2,3),\ (3,4) \}$

(e) $\{ (1,1),\ (2,2),\ (3,3),\ (3,4) \}$

(f) $\{ (1,3),\ (1,4),\ (2,3),\ (2,4),\ (3,1),\ (3,4) \}$

2. Determine whether the relations described by the conditions below are reflexive, symmetric or transitive, and give reasons for your responses.

(a) All ordered pairs of real numbers (x, y) such that xy is rational.

(b) All ordered pairs of real numbers (x, y) such that $x = 2y$.

(c) All ordered pairs of real numbers (x, y) such that $xy \geq 0$.

(d) All ordered pairs of real numbers (x, y) such that $xy = 0$.

3. Determine whether the relations described by the conditions below are reflexive, symmetric or transitive, and give reasons for your responses.

(a) All ordered pairs of real numbers (x, y) such that $x \neq y$.

(b) All ordered pairs of real numbers (x, y) such that $xy \geq 1$.

(c) All ordered pairs of real numbers (x, y) such that $x = y \pm 1$.

(d) All ordered pairs of real numbers (x, y) such that $x = y^2$.

(e) All ordered pairs of real numbers (x, y) such that $x \geq y^2$.

4. Which of the relations described below on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a) All **a** and **b** such that **a** and **b** have the same age.

(b) All **a** and **b** such that **a** and **b** have the same parents.

(c) All **a** and **b** such that **a** and **b** have a common parent.

(d) All **a** and **b** such that **a** and **b** have met.

(e) All **a** and **b** such that **a** and **b** speak a common language.

5. Let $\mathcal{R}$ be a binary relation that is reflexive and transitive, and define a new binary relation $\mathcal{S}$ such that $x\mathcal{S}y$ if and only if $x\mathcal{R}y$ and $y\mathcal{R}x$. Prove that $\mathcal{S}$ is an equivalence relation.

6. A binary relation $\mathcal{R}$ is said to be **circular** if it satisfies $[a\mathcal{R}b$ and $b\mathcal{R}c \;\Rightarrow\; c\mathcal{R}a]$. Show that $\mathcal{R}$ is an equivalence relation if and only if it is reflexive and circular.

7. Let $\mathbb{R}$ denote the real numbers, and let $\mathcal{S}$ be the binary relation on $\mathbb{R} - \{0\}$ such that $(x, y)\mathcal{S}(z, w)$ if and only if $xw = yz$. Prove that $\mathcal{S}$ is an equivalence relation, and show that every equivalence class contains a unique element (or representative) of the form $(r, 1)$.

8. Suppose that S is a finite set, and given a partition $C_1, \cdots, C_r$ of S suppose that C_j contains k_j elements for each j. If S has n elements, then the different types of partitions for S can be listed systematically by sequences of positive integers $k_1, \cdots, k_r$ where $r = 1, \ldots n$, $k_1 \geq \cdots k_r$ and $\sum k_i = n$. Find the number of partition types if S is a set with 5 elements.

9. Let $A = 1, 2, 3, 4, 5, 6$ and consider the equivalence relation $\mathcal{E}$ on A such that the related pairs are $(1, 1)$, $(1, 5)$, $(2, 2)$, $(2, 3)$, $(2, 6)$, $(3, 2)$, $(3, 3)$, $(3, 6)$, $(4, 4)$, $(5, 1)$, $(5, 5)$, $(6, 2)$, $(6, 3)$ and $(6, 6)$. Find the equivalence classes of $\mathcal{E}$.

10. Let $\mathcal{R}_1$ be the relation a evenly divides b on the positive integers, and let $\mathcal{R}_2$ be the relation a is an integral multiple of b on the positive integers. Describe the relations $\mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathcal{R}_1 \cap \mathcal{R}_2$.