Binary relations and equivalence

Cartesian products provide a solid mathematical framework for stating that two objects are somehow related.

Definition. If A and B are two sets, then a ***binary relation*** from A to B is a subset $\mathfrak{R}$ of $A \times B$. We shall often say that x _is_ $\mathfrak{R}-$ _related to_ y or that x _is in the_ $\mathfrak{R}-$ _relation to_ y if $(x, y) \in \mathfrak{R}$. Frequently we shall also write $x\,\mathfrak{R}\,y$ to indicate this relation holds for x and y in that order.

If $A = B$ then a binary relation from A to A is simply called a ***binary relation on*** A.

Some of these abstractly defined binary relations are not particularly interesting. In particular, both the empty set and all of $A \times B$ satisfy the condition to be a binary relation, but neither carries any new information distinguishing one ordered pair (a, b) from another (a', b'). Another less trivial, but still relatively unenlightening, example of a binary operation on an arbitrary set A is given by the _diagonal relation_ Δ_A consisting of all ordered pairs (x, y) such that $x = y$. If $\mathfrak{R} = \Delta_A$ then $x\,\mathfrak{R}\,y$ simply means that x and y are equal.

We clearly need more substantial examples to justify the definition of a binary relation.

Example 1. Let A be the integers, rational numbers or real numbers, and take the binary relation on A consisting of all (x, y) such that $x \leq y$.

Example 2. Let A be the integers, and take the binary relation on A consisting of all ordered pairs (x, y) such that $x - y$ is even. In this case x and y are related if and only if either both are even or both are odd.

Example 3. Let A be the **positive** integers, and take the binary relation on A consisting of all pairs (x, y) such that the quotient y/x is a positive integer (in other words, x evenly divides y with no remainder).

Example 4. In this example A will correspond to the squares on a chessboard, so that we can identify A with

$$\{ 1, 2, 3, 4, 5, 6, 7, 8 \} \times \{ 1, 2, 3, 4, 5, 6, 7, 8 \}$$

and (x, y) will be related to (x', y') if and only if one of the two quantities $|x - x'|$ and $|y - y'|$ is equal to 1 and the other is equal to 2. In nonmathematical terms this relation corresponds to the condition in chess that a knight positioned at square (x, y) is able to reach square (x', y') in one move, assuming that the latter is not occupied by a piece of the same color.

Example 5. In this example let A be the set of all polynomials with real coefficients, and stipulate that a polynomial $f(t)$ is related to $g(t)$ if there is a third polynomial $P(x)$ such that $g(t) = P(f(t))$.

Example 6. This is given by the rock – paper – scissors game. Let A be the set {**rock, paper, scissors**}, and stipulate that object x is related to object y if object x wins over y under the usual rules of the game (scissors wins over paper, while paper wins over rock, and rock wins over scissors).

Example 7. In this example A and B will be distinct sets. Take A to be the plane (either in the classical Euclidean sense or the coordinate sense), let $B = A \times A$, and consider the relation $z \,\mathfrak{R}\, (x, y)$ if and only if x and y are distinct and z lies on the (unique) line determined by x and y.

Example 8. Given a set A, take the binary relation on the set $\mathcal{P}(A)$ of all its subsets defined by $B \,\mathfrak{R}\, C$ if and only if B is a subset of C.

Here is a chart for the last example if A consists of two elements. We may index the elements of $\mathcal{P}(A)$ by two digit numbers qr where each of q and r is either 0 or 1 (hence 10 corresponds the subset which contains q but not r, 11 corresponds the subset which contains everything, and 00 corresponds to the empty set). In the chart the first coordinate of (B, C) is given by the row and the second by the column; a square is marked by a plus sign if B is contained in C and by a circle otherwise. Setting up a similar chart for a set with three elements is a strongly recommended exercise.

$C \backslash B$	00	01	10	11
00	+	O	O	O
01	+	+	O	O
10	+	O	+	O
11	+	+	+	+

Equivalence relations

Frequently in mathematics it is important to understand whether two objects have some common properties even if they might not be identical. This is made precise in the concept of an equivalence relation.

Definitions. Let $\mathfrak{R}$ be a binary relation on a set A.

$\mathfrak{R}$ is ***reflexive*** if $a \,\mathfrak{R}\, a$ for all $a \in A$.

$\mathfrak{R}$ is ***symmetric*** if $a \,\mathfrak{R}\, b$ implies $b \,\mathfrak{R}\, a$ for all $a, b \in A$.

$\mathfrak{R}$ is ***transitive*** if $a \,\mathfrak{R}\, b$ and $b \,\mathfrak{R}\, c$ imply $a \,\mathfrak{R}\, c$ for all $a, b, c \in A$.

Observe that the equality relation $a = b$ satisfies all three of these conditions. More generally, we say that $\mathfrak{R}$ is an ***equivalence relation*** if it satisfies all of the three properties defined above. As noted in Cunningham, an equivalence relation $a \,\mathfrak{R}\, b$ is frequently denoted by $a \sim b$.

An arbitrary binary relation on a set might or might not sarisfy some or all of these properties. In fact, for each subset of this list there are binary relations which satisfy the properties in that subset but do not satisfy any of the others (in other words, the conditions are ***logically independent*** or each other). Here is a chart indicating whether a given property is valid for the examples given above:

property\example	1	2	3	4	5	6	8
reflexive	+	+	+	O	+	O	+
symmetric	O	+	O	+	O	O	O
transitive	+	+	+	O	+	O	+

In particular, only the second example is an equivalence relation. Here is another example with all the details worked out:

Example 9. If A is the set of real numbers, consider the binary relation $a \,\mathfrak{R}\, b$ if and only if $b - a$ is an integer. Then $\mathfrak{R}$ is an equivalence relation.

VERIFICATION. The reflexive law is valid because $a - a = 0$ and 0 is an integer. Furthermore, the symmetric law is also valid, for if $b - a$ is the integer n then $a - b$ is the integer $-n$. Finally, the transitive law is also valid, for if $b - a$ is the integer n and $c - b$ is the integer m, then $c - a = (c - b) + (b - a)$ is the integer $m + n$. ■

Equivalence classes and partitions. If two objects in the set A are related by an equivalence relation, it generally means that they have certain properties in common. Given $a \in A$ and an equivalence relation $\mathcal{E}$ on A, it is natural to consider all members of A which have a given common property. The remainder of this lecture is devoted to considering such subsets of A.

Definition. If A is a set, $a \in A$, and $\mathcal{E}$ is an equivalence relation on A, then the $\mathcal{E}-$ ***equivalence class*** of a, written $[a]_{\mathcal{E}}$ or simply $[a]$ if $\mathcal{E}$ is clear from the context, is the set of all $x \in A$ such that x is $\mathcal{E}-$ related to a. — If C is an equivalence class for $\mathcal{E}$ and $x \in C$, then one frequently says that x _is a representative for the equivalence class_ C (or something that is grammatically equivalent).

Since equivalence classes for $\mathcal{E}$ are subsets of A, we have the following elementary observation.

Proposition. *If A is a set and $\mathcal{E}$ is an equivalence relation on A, then the collection of all $\mathcal{E}-$ equivalence classes is a set.*

PROOF. By construction the collection of all equivalence classes is a subcollection of the set $\mathcal{P}(A)$.■

The set of all equivalence classes is often denoted by symbolism such as $A/\mathcal{E}$, and it is often verbalized as "A **modulo** $\mathcal{E}$" or (more briefly) "A **mod** $\mathcal{E}$."

Equivalence classes for previous examples. In Example 2, the equivalence class of an integer a is the set of all even integers if a is even and the set of all odd integers if a is odd. For the equality relation(s), the equivalence class of a is simply the set $\{a\}$. In the example involving real numbers(Example 9), the equivalence class of a number is the set of all real numbers such that the decimal expansions to the right of the decimal point are the same.

The equivalence classes of an equivalence relation have the following fundamentally important property:

Theorem. *Let A be a set, suppose that x and y belong to A, and let $\mathcal{E}$ be an equivalence relation on A. Then either the equivalence classes $[x]_{\mathcal{E}}$ and $[y]_{\mathcal{E}}$ are disjoint or else they are equal.*

PROOF. Suppose that the equivalence classes in question are not disjoint, and let z belong to both of them. Then we have $x\ \mathcal{E}\ z$ and $y\ \mathcal{E}\ z$. By symmetry, the second of these implies $z\ \mathcal{E}\ y$, and one can combine the latter with $x\ \mathcal{E}\ z$ and transitivity to conclude that $x\ \mathcal{E}\ y$.

Suppose now that $w \in [y]_{\mathcal{E}}$ so that $y\ \mathcal{E}\ w$. By transitivity and the final conclusion of the previous paragraph it follows that $y\ \mathcal{E}\ w$, so that $w \in [x]_{\mathcal{E}}$ is also true. Therefore we have shown that $[y]_{\mathcal{E}} \subset [x]_{\mathcal{E}}$. If we reverse the roles of x and y in this argument and note that $x\ \mathcal{E}\ y$ implies $y\ \mathcal{E}\ x$, we can also conclude that $[x]_{\mathcal{E}} \subset [y]_{\mathcal{E}}$. Combining this with the preceding sentence, we have $[y]_{\mathcal{E}} = [x]_{\mathcal{E}}$.■

Corollary (Partition Property). *The equivalence classes of an equivalence relation on A form a family of pairwise disjoint subsets whose union is all of A.*■

A converse to the preceding corollary also plays an important role in the study of equivalence relations:

<u>Proposition.</u> *Let A be a set, and let* **C** *be a family of subsets of A such that (i) the subsets in* **C** *are pairwise disjoint, (ii) the union of the subsets in* **C** *is equal to A. Then there is an equivalence relation $\mathcal{E}$ on A whose equivalence classes are the sets in the family* **C**.

The family **C** is said to define a ***<u>partition</u>*** of the set A.

<u>PROOF.</u> We define a binary relation $\mathcal{E}$ on A by stipulating that $x\ \mathcal{E}\ y$ if and only if there is some $B \in$ **C** such that $x \in B$ and $y \in B$.

Our first objective is to prove that $\mathcal{E}$ is an equivalence relation. To see that $x\ \mathcal{E}\ x$ for all x, let x be arbitrary and use the hypothesis that the union of the subsets in **C** is A to find some set B such that $x \in B$. We then have $x \in B$ and $x \in B$, and therefore it follows that $x\ \mathcal{E}\ x$.

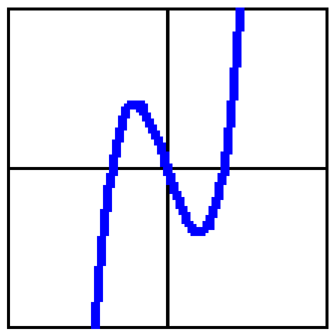
Suppose now that $x\ \mathcal{E}\ y$, so that there is some $B \in$ **C** such that $x \in B$ and $y \in B$. We then also have $x \in B$ and $x \in B$, and therefore it follows that $y\ \mathcal{E}\ x$.

Finally, suppose that $x\ \mathcal{E}\ y$ and $y\ \mathcal{E}\ z$. Then by the definition of $\mathcal{E}$ there are subsets $B, D \in$ **C** such that $x \in B$ and $y \in B$ and also $y \in D$ and $z \in D$. It follows that B and D have y in common, and since the family **C** of subsets is pairwise disjoint, it follows that the subsets B and D must be equal. But this means that $x \in B$, $y \in B$ and $z \in B$. Therefore we have $x\ \mathcal{E}\ z$, and this completes the proof that $\mathcal{E}$ is an equivalence relation.

What is the equivalence class of an element $x \in A$? Choose B such that $x \in B$; since B is the ***unique*** subset from the family **C** that contains x, it follows that $x\ \mathcal{E}\ y$ if and only if y also belongs to B. Therefore B is the equivalence class of x. Since x was arbitrary, it follows that the equivalence *classes* of $\mathcal{E}$ are just the subsets in the original family **C**.■

<u>Example 10.</u> Among other things, the following example illustrates how different equivalence classes can have different numbers of elements. Let A be the real numbers, and consider the relation $x\ \mathcal{R}\ y$ if and only if $x^3 - 27x = y^3 - 27y$. It is fairly straightforward to verify that this defines an equivalence relation on the real numbers, and the equivalence classes consist of all values of x such that $x^3 - 27x$ is equal to a specific real number a.

One way to visualize the equivalence classes of $\mathfrak{R}$ is to take the graph of $x^3 - 27x$ and look at its intersection with a fixed horizontal line of the form $y = a$. If we sketch of the graph for $y = x^3 - 27x$ as in the picture below, it is apparent that for some choices of a one obtains equivalence classes with one point, for exactly two choices of a the equivalence classes consist of two points, and for still other choices of a the equivalence classes consist of three points.



The cases with two points occur when the tangent line to the graph is horizontal, which happens when $|x| = 3$, and hence when $|a| = 54$. Thus equivalence classes have exactly one element if $|a| < 54$, exactly two elements if $|a| = 54$, and exactly three elements if $|a| > 54$.