Generating equivalence relations

Given a binary relation $\mathcal{R}$ on a set A, there are numerous situations where one wants to describe an equivalence relation $\mathcal{E}$ such that $x \; \mathcal{E} \; y$ if x and y are $\mathcal{R}-$ related. By the definition of a binary relation, this amounts to saying that $\mathcal{R}$ is contained in $\mathcal{E}$ as a subset of $A \times A$. The following result shows that every binary relation $\mathcal{R}$ is contained in a *unique minimal* equivalence relation:

Theorem (Generating an equivalence relation). *Let A be a set, and let $\mathcal{R}$ be a binary relation on A. Then there is a unique minimal equivalence relation $\mathcal{E}$ such that $\mathcal{R} \subset \mathcal{E}$.*

PROOF. Define a new binary relation $\mathcal{E}$ on A so that $x \; \mathcal{E} \; y$ if and only if there is a finite sequence of elements of A

$$x = x_1, \ldots, x_n = y$$

such that for each $k = 1, \ldots, n$ one (or more) of the following holds:

$$x_k = x_{k+1}$$
$$x_k \; \mathcal{R} \; x_{k+1}$$
$$x_{k+1} \; \mathcal{R} \; x_k$$

Suppose that $\mathcal{F}$ is an equivalence relation that contains $\mathcal{R}$ and that $x \; \mathcal{E} \; y$. Then for each k it follows that $x_k \; \mathcal{F} \; x_{k+1}$, and therefore by repeated application of transitivity it follows that $x \; \mathcal{F} \; y$. Therefore, if $\mathcal{E}$ is an equivalence relation it will follow that it must be the unique minimal equivalence relation containing $\mathcal{R}$.

To prove that $\mathcal{E}$ is reflexive, for each $x \in A$ it suffices to consider the simple length two sequence x, x and notice that the first option then guarantees that $x \; \mathcal{E} \; x$. Suppose now that $x \; \mathcal{E} \; y$, and take a sequence

$$x = x_1, \ldots, x_n = y$$

as before. If we define a new sequence

$$y = y_1, \ldots, y_n = x$$

where $y_p = x_{n+1-p}$ (reverse the order of the terms), then by the assumption on the original sequence we know that one of $y_p = y_{p+1}$, $y_p \; \mathcal{R} \; y_{p+1}$ or $y_{p+1} \; \mathcal{R} \; y_p$ holds. Therefore $y \; \mathcal{E} \; x$, and hence the relation $\mathcal{E}$ is symmetric.

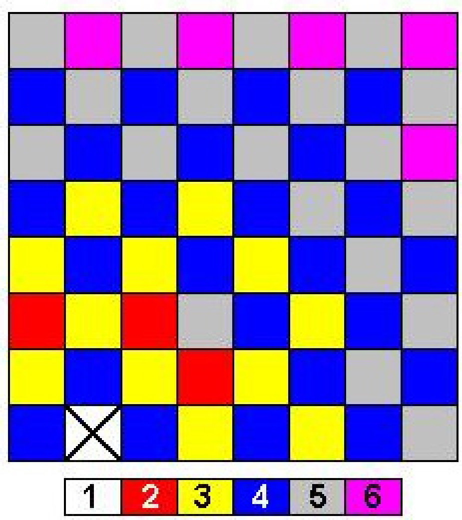
Finally, suppose that $x\ \mathcal{E}\ y$ and $y\ \mathcal{E}\ z$. Then we have two sequences

$$x = x_1, \ldots, x_n = y \quad \text{and} \quad y = y_1, \ldots, y_n = z$$

such that consecutive terms satisfy one of the three conditions listed above. Therefore if we define a new sequence whose terms w_p are given by w_p if $p \leq n$ and by y_{p-n+1} if $p > n$, it will follow that consecutive terms satisfy one of the three conditions we have listed. This means that $\mathcal{E}$ is transitive and hence is an equivalence relation.■

<u>Example.</u> Suppose we take Example $\mathbf{2}$ from Lecture 06, in which the binary relation $\mathcal{R}$ models the possible moves for a knight in the game of chess. Chess books frequently claim that a knight can reach any square on the board after a finite number of moves. We can translate this into mathematics as follows: *If $\mathcal{E}$ is the equivalence relation generated by $\mathcal{R}$, then every pair of points is related by $\mathcal{E}$.*

<u>VERIFICATION.</u> By definition we know that $\mathcal{R}$ is symmetric (one can always reverse a move of a piece that is not a pawn), and $\mathcal{R}$ is also reflexive by definition. Therefore the proof amounts to showing that one can get from a given square to any other square by a finite sequence of moves. The drawing below indicates how this can be done if the initial position is the usual starting position for a knight on the board.



We shall explain the meaning of this picture and leave the verification of the statement about E to the reader. The color code indicates when a knight will get from the original position marked with an **X** to a given square on the chessboard. For example, the possible positions after one move are colored red, and the possible positions after two moves are colored yellow. Note that the knight can reach every square in at most five moves. It also follows that if the knight starts at an arbitrary square, then it can reach any other square within finitely many moves. You might want to figure out the minimum number of moves from any given square to another arbitrary square.