Math 144
Fall 2022

SOLUTIONS FOR WEEK 03 EXERCISES

GENERAL REMARK. There are several exercises which ask whether a given binary relation is reflexive, symmetric, antisymmetric or transitive. We shall only work out a few representative examples in detail and give yes/no answers for the others.

1. (a) This relation is not reflexive because $(1,1)$ and $(4,4)$ are not elements of the subset. It is not symmetric because it contains $(2,4)$ but not $(4,2)$. To see it is transitive, one needs to enumerate all the pairs of ordered pairs (a,b) and (b,c) in the relation:

[1] $(2,2)$, $(2,2)$
[2] $(2,2)$, $(2,3)$
[3] $(2,2)$, $(2,4)$
[4] $(2,3)$, $(3,2)$
[5] $(2,3)$, $(3,3)$
[6] $(2,3)$, $(3,4)$
[7] $(3,2)$, $(2,2)$
[8] $(3,2)$, $(2,3)$
[9] $(3,2)$, $(2,4)$
[10] $(3,3)$, $(3,3)$
[11] $(3,3)$, $(3,4)$

The transitivity of this relation amounts to saying that for each of these cases the corresponding ordered pair (a,c) lies in the relation. One checks this out on a case by case basis.■

(b) This relation is reflexive, symmetric and transitive. We shall only give details for the first two because the others are worked in a manner similar to the previous exercise. The relation is reflexive because it contains each ordered pair (x,x). To show it is symmetric, one must list all the ordered pairs in the relation

$$(1,1),\ (2,2),\ (2,1),\ (1,2),\ (3,3),\ (4,4)$$

and check that the pairs with the entries switched

$$(1,1),\ (2,2),\ (1,2),\ (2,1),\ (3,3),\ (4,4)$$

also belong to the relation, which is straightforward.■

(c) This relation is symmetric, but not reflexive or transitive. We have already done examples for the first two types, so we shall only give details for the last conclusion. This follows because the relation contains $(2,4)$ and $(4,2)$ but neither $(2,2)$ nor $(4,4)$. If a relation is transitive and contains both $(2,4)$ and $(4,2)$, then it must also contain both $(2,2)$ and $(4,4)$.■

(d) This relation is not reflexive, symmetric or transitive. The latter is **vacuously true** because the relation does not contain two ordered pairs of the form (a,b) and (b,a)!■

(e) This relation is not reflexive, symmetric, or transitive.■

(f) This relation is not reflexive, symmetric, or transitive.■

2. (a) This relation is reflexive, symmetric and transitive.■

(b) This relation is not reflexive, not symmetric and not transitive. We shall only check the last of these. If xRy and yRx then we have $x = 2y$ and $y = 2x$. The only way this can happen is if $x = y = 0$, and of course it does happen in this case.■

(c) This relation is reflexive and symmetric, but it is not transitive.■

(d) This relation is symmetric, but it is not reflexive and not transitive.■

3. (a) This relation is symmetric, but it is not reflexive and not transitive.■

(b) This relation is symmetric, but it is not reflexive and not transitive.■

(c) This relation is symmetric, but it is not reflexive and not transitive.■

(d) This relation is not reflexive, not symmetric and not transitive.

(e) This relation is not reflexive, not symmetric, and not transitive. We shall only check the last two of these. A counterexample to transitivity is given by $(x,y) = (\frac{1}{3}, \frac{1}{2})$ and $(y,z) = (\frac{1}{2}, \frac{3}{5})$. For these choices we have $x > y^2$ and $y > z^2$ but $x < z^2$.■

4. (a) Equivalence relation.■

(b) Equivalence relation.■

(c) Not transitive.■

(d) Not transitive.■

(e) Not transitive.■

5. The relation S is reflexive, for R is reflexive and xRx and xRx imply xSx. Suppose now that xSy. Then xRy and yRx, and by definition this also implies ySx. Finally, suppose that xSy and ySz. Then we have xRy and yRx and, we also have yRz and zRy. By the transitivity of R these imply that xRz and zRx, which means that xSz. Therefore S is an equivalence relation.■

6. We first prove the ($\Longrightarrow$) implication. Suppose that R is an equivalence relation. Then it is automatically reflexive. Suppose now that xRy and yRz. Then we also have xRz because R is transitive. But since R is symmetric the latter implies zRx and hence T is circular. Now we prove the ($\Longleftarrow$) implication. By assumption R is reflexive. To show that it is symmetric, suppose that xRy. If we combine this with xRx (since R is reflexive) and the circular property we conclude that yRx. Finally, if xRy and yRz, then zRx since

R is circular. However, we have shown that R is symmetric, and therefore we also have xRz so that R is transitive. Hence R is an equivalence relation.$\blacksquare$

7. We have $(x, y)P(x, y)$ because $xy = xy$, and if $(x, y)P(z, w)$ then $xw = yz$, which is equivalent to $zy = wx$, which means that $(z, w)P(x, y)$. Finally, if $(x, y)P(z, w)$ and $(z, w)P(u, v)$, then $xw = yz$ and $zv = uw$. Multiplying these equations together yields $xwzv = yzuw$, and since $w \neq 0$ it follows that $xzv = yzu$. If $z \neq 0$ then we may divide both sides of the equation by z and obtain $xv = yu$, which implies $(x, y)P(u, v)$.

Suppose now that $z = 0$. Then $0 = yz = xw$ and since $w \neq 0$ it follows that $x = 0$. Likewise, $0 = zv = uw$ implies $u = 0$. But then we have $xv = 0v = 0 = 0w = uw$, so that $(x, y)P(u, v)$ in this case too. Therefore we have shown that the relation is an equivalence relation.

To prove the final assertion, we first show there is at least one r such that $(x, y) = (r, 1)$. Specifically, if $r = x/y$, then $yr = x = x \cdot 1$. Next, we must show there is only one such r, so suppose we have $(x, y)P(s, 1)$. Then by the definition of the equivalence relation we have $ys = x$, so that $s = x/y$, and hence s must be equal to the value of r give previously.$\blacksquare$

8. We shall list the partitions using a **greedy algorithm:** The first subset will have the largest possible number of elements if we exclude the partition types already listed, then the second will also have the largest possible number of elements if we exclude previously listed types, and so on. Here is what we get:

5
$4 \geq 1$
$3 \geq 2$
$3 \geq 1 \geq 1$
$2 \geq 2 \geq 1$
$2 \geq 1 \geq 1 \geq 1$
$1 \geq 1 \geq 1 \geq 1 \geq 1$ $\blacksquare$

9. The equivalence classes are $\{1, 5\}$, $\{4\}$, and $\{2, 3, 6\}$.$\blacksquare$

10. The union of the relations is that a is a multiple of b or b is a multiple of a, and the intersection is that a is a multiple of b and b is a multiple of a. The first of these cannot be simplified, but the second can as follows: if $a = xb$ and $b = ya$, then $a = xb = xya$ implies that $xy = 1$, so that $x = y = \pm 1$, and hence the intersection is the relation that $a = \pm b$.$\blacksquare$