Math 144
Winter 2022

SOLVED PROBLEMS FOR WEEK 03

1. Consider the binary relation on $\{1, 2, 3, 4\}$ given by $\mathcal{R} = \{ (1,1), (2,2), (2,3), (3,2), (4,4) \}$. Determine whether this relation is reflexive, symmetric or transitive, and give reasons for your responses.

SOLUTION.

The relation is not reflexive because 3 is not related to itself. The relation is symmetric, for if we switch the second and first coordinates for every element in $\mathcal{R}$ we get back $\mathcal{R}$. The relation is not transitive because $3\mathcal{R}2$ and $2\mathcal{R}3$ are true but $3\mathcal{R}3$ is not.■

2. Determine whether the relations described by the conditions below are reflexive, symmetric or transitive, and give reasons for your responses.

(a) All ordered pairs of nonnegative integers (x, y) such that $x > y$.

(b) All ordered pairs of positive integers (x, y) such that xy is a perfect square (of some other positive integer).

(c) All ordered pairs of nonnegative integers (x, y) such that $x + y = 10$.

(d) All ordered pairs of nonnegative integers (x, y) such that $x + 4y = 10$.

SOLUTION.

(a) The relation is not reflexive because a nonnegative integer is never greater than itself. The relation is not symmetric because x can never be both greater than and less than y. The relation is transitive because $x > y$ and $y > z$ always implies $x > z$.■

(b) The relation is reflexive because every positive integer of the form x^2 has a positive square root (namely, x). Furthermore, the relation is symmetric because $xy = yx$, so that xy is a perfect square if and only if yx is. Finally, the relation is also transitive; the argument requires the result that every positive integer is a product of primes:

Let $p_1, \cdots, p_n$ be all the primes which are factors of x, y or z, and write

$$x = p_1^{a_1} \cdots p_n^{a_n} , \qquad y = p_1^{b_1} \cdots p_n^{b_n} , \qquad z = p_1^{c_1} \cdots p_n^{c_n}$$

where an exponent is zero if the corresponding prime does not divide the number on the left.

Suppose now that xy and yz are perfect squares. Then all of the exponents $a_k + b_k$ and $b_k + c_k$ are even, and therefore all the sums $a_k + 2b_k + c_k$ are even. But this means that all the sums $a_k + c_k$ are also even, which implies that xz is also a perfect square and hence the relation is transitive.■

(c) The relation is not reflexive because the equation $x + x = 10$ is valid if and only if $x = 5$, so in particular 4 is not related to itself. The relation is symmetric because $x + y = 10$ is valid if and only if $x + y = 10$ is valid. The relation is not transitive because $x + y = 10$ and $y + z = 10$ imply $x = z$, and we know this is not true unless $x = z = 5$.■

(d) The relation is not reflexive because $x + 4x = 10$ is valid if and only if $x = 2$. Furthermore, the relation is not symmetric because $x + 4y = 10$ is true for nonnegative integers x and y if and only if $(x, y) = (10, 0)$, $(6, 1)$ or $(2, 2)$ while $y + 4x = 10$ if and only if $(x, y) = (0, 10)$, $(1, 6)$ or $(2, 2)$. Finally, the relation is transitive; this follows because the only way that pairs of the form (x, y) and (y, z) can be related is if $y = 2$, in which case $x = y = z = 2$. Since the pair $(x, z) = (2, 2)$ is related, it follows that the relation is transitive.■

3. Find the mistake in the following argument which claims to show that the symmetric and transitive properties imply the reflexive property: Given x, take y such that $x\mathcal{R}y$ is true. By symmetry we also know that $y\mathcal{R}x$ is true, and thus by transitivity we also know that $x\mathcal{R}y$ is true. Find a simple, noncircular way to modify the hypothesis so that the argument becomes valid. (A circular argument is one in which the conclusion is part of the hypothesis.)

SOLUTION.

The hypotheses do not guarantee that an arbitrary element of the set is related to anything. One easy way to repair the result is to add an assumption that every x in the set is related to some element (but **not necessarily itself**, for that would make the argument circular!).■