EXERCISES FOR WEEK 04

0. Work the following problems from Cunningham:

Exercises 3.3 (pp. 68–69): 2, 7, 8, 11, 15
Exercises 3.5 (pp. 81–82): 1, 8

1. For each of the examples below. determine whether the set in question is the graph of a function on the real line $\mathbb{R}$ and give reasons for your answers.

(a) The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $x^2 + y^2 = 1$.

(b) The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $x^2 + y = 1$.

(c) The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $xy = 1$.

(d) The set of all points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $xy = 0$.

2. Let P be the set of all U. S. presidents, and let G be the set of all ordered pairs $(a, b) \in P \times P$ such that b succeeded a in office; a president is not considered to succeed himself if he was re-elected while in office. Is G the graph of a function? Explain your answer.

3. Let A and x be sets. Prove that there is a unique function from A to $\{x\}$. It might be helpful to split the proof into two cases depending upon whether or not A is empty.

4. Prove that for each set X there is a unique function from the empty set to X, regardless of whether or not X is nonempty. Also prove that there are no functions from X to the empty set if X is nonempty.

5. Find the image of the function which assigns to each positive integer the number of digits 1, 2, 3, 4, 5, 6, 7, 8, 9 that do not appear in the base 10 decimal expansion of the integer. Give reasons for your answer.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 3x - 7$. Compute the following sets:

(a) $f^{-1}\,[\,[-7, 2]\,]$

(b) $f\,[\,[2, 6]\,]$

7. Let A be the set $\{1, 2, 3, 4, 5\}$, and let $f : A \to A$ be the function defined by $f(1) = 3$, $f(2) = 5$, $f(3) = 5$, $f(4) = 2$ and $f(5) = 3$. Find the following subsets:

(a) $f[A]$

(b) $f[S]$ if $S = \{1, 2, 3\}$

(c) $f^{-1}[S]$ if $S = \{1, 2, 3\}$

8. Let $f(x) = 2x + 1$ and $g(x) = x^2 - 2$. Find $f \circ g(x)$ and $g \circ f(x)$.

9. Suppose that

$$f(x) = \frac{x-2}{x-3}$$

Find the largest possible set of real numbers E such that f defines a 1–1 onto mapping from E to itself, and give an explicit formula for f^{-1}.

10. Suppose that $f : A \to B$ and $g : B \to C$ are functions. Prove the following:

(a) If $g \circ f$ is 1–1, then so is f.

(b) If $g \circ f$ is onto, then so is g.

11. Given a set X, let $\mathcal{P}_+(X)$ denote the nonempty subsets of X. Suppose now that A and B are sets. Show that the mapping $h : \mathcal{P}_+(A) \times \mathcal{P}_+(B) \to \mathcal{P}_+(A \times B)$ which sends (C, D) to $C \times D \subset A \times B$ is 1–1, and give an example to show it is not necessarily onto.

12. Determine whether each of the following functions from the set $\{a, b, c, d\}$ to itself is 1–1.

(a) The function sending the ordered quadruple (a, b, c, d) to (b, a, c, d).

(b) The function sending the ordered quadruple (a, b, c, d) to (b, b, d, c).

13. Determine which of these functions are 1–1 and onto from the set of real numbers to itself.

(a) $f(x) = -3x + 4$.

(b) $f(x) = -3x^2 + 7$.

(c) $f(x) = (x + 1)/(x + 2)$.

(d) $f(x) = x^5 + 1$.

14. Let X and Y be nonempty sets, and let $f : X \to Y$ be a function.

(a) Prove that $f[A \cap B] = f[A] \cap f[B]$ for all subsets A and B of X if and only if f is 1–1.

(b) Prove that $f[X - A] \subset Y - [A]$ for all subsets A of X if and only if f is 1–1.

(c) Prove that $Y - f[A] \subset f[X - A]$ for all subsets A of X if and only if f is onto.

15. Let $[0, 1]$ be the closed unit interval, and let a and b be real numbers which satisfy a$< b$. Construct a bijection from $[0, 1]$ to $[a, b]$. Is it unique? [*Hint:* For the second part, consider the case where $[a, b] = [0, 1]$.]

16. Find the inverse function to $f(x) = x/(1+|x|)$, where the domain and the codomain are the real numbers. [*Hint:* In the second example it is useful to consider two cases depending upon whether $x \geq 0$ or $x \leq 0$.]

17. Let A, B and C be sets. Prove that there is a 1–1 correspondence between functions $h : C \to A \times B$ and ordered pairs of functions $(f : A \to B,\ g : B \to C)$ defined by $h(x) = (\,f(x),\ g(x)\,)$.

18. Let A and B be sets, and let $T_{A,B} : A \times B \to B \times A$ be the transposition map sending (a, b) to (b, a). Prove that $T_{A,B}$ is a 1–1 onto mapping and its inverse function is given by $T_{B,A}$.

19. (a) Let A and B be nonempty sets, and let $a \in A$. Prove that the slice embedding $j : B \to A \times B$ defined by $j(b) = (a, b)$ is a 1–1 mapping and that $\pi_B \circ j$ is a 1–1 onto mapping.

(b) If $\varphi : A \times B \to A \times B$ is the composite $j \circ \pi_B$, what is $\varphi \circ \varphi$?

20. Let $f : A \to B$ be a function, where A and B are nonempty. Verify that $f = Q \circ J$, where $Q : A \times B \to B$ is coordinate projection and $J : A \to A \times B$ sends $a \in A$ to $(a,\ f(a)\,)$. Explain why J is 1–1 and Q is onto.