Functions – II

This is a continuation from the previous lecture, starting with more properties of the images and inverse images of functions.

Composition, images and inverse images. The image and inverse image constructions are highly compatible with composition of functions.

Proposition. *Suppose that $f : A \to B$ and $g : B \to C$ are functions, and let M and N denote subsets of A and C respectively. Then we have*

$$g \circ f\,[M] \;=\; g\big[\,f[M]\,\big] \qquad \textit{and} \qquad (g \circ f)^{-1}[N] \;=\; f^{-1}\big[\,g^{-1}[N]\,\big].$$

Proof. We shall first verify that $g \circ f\,[M] = g\big[\,f[M]\,\big]$. Suppose that $z = g \circ f(x)$ for some $x \in M$. Since $g \circ f(x) = g(f(x))$ it follows that we have $z = g(y)$ where $y = f(x)$ and $x \in M$. Therefore $y \in f[M]$ and consequently we must also have $z \in g\big[\,f[M]\,\big]$. To prove the reverse inclusion, suppose that $z \in g\big[\,f[M]\,\big]$, so that $z = g(y)$ where $y = f(x)$ and $x \in M$. We may then use $g \circ f(x) = g(f(x))$ to conclude that $y \in g \circ f\,[M]$, completing the proof of the second inclusion and thus also the proof that the two sets under consideration are equal.

We shall next verify that $(g \circ f)^{-1}[N] = f^{-1}\big[\,g^{-1}[N]\,\big]$. Suppose that x belongs to the set $(g \circ f)^{-1}[N]$. By definition we have $g \circ f(x) \in N$, and since $g \circ f(x) = g(f(x))$ it follows that $f(x) \in g^{-1}[N]$. The latter in turn implies that $x \in f^{-1}\big[\,g^{-1}[N]\,\big]$, and this proves containment in one direction. To prove containment in the other direction, suppose that $x \in f^{-1}\big[\,g^{-1}[N]\,\big]$. Working backwards, we see that $f(x) \in g^{-1}[N]$, so that $g \circ f(x) = g(f(x)) \in N$, which implies that $x \in (g \circ f)^{-1}[N]$. This proves containment in the other direction and hence that the two sets under consideration are equal.■

Inclusion functions and restrictions to subsets. If C is a subset of A, then there is an ***inclusion function*** $i_{C \subset A} : C \to A$ whose graph is the diagonal set of all (x, x) where $x \in C$. One reason for introducing this function is that there are some constructions on functions **K** and inclusions $C \subset A$ such that **K(C)** is not necessarily a subset of **K(A).** Another reason is that it gives a formal basis for restricting a function to a subset. Specifically, if $f : A \to B$ is a function and C is a subset of A, then the *restriction of* f *to* C is the composite function $f \circ i_{C \subset A} : C \to B$; this restricted function is generally denoted by $f \mid C$. If the graph of f is the set $G \subset A \times B$, then the graph of $f \mid C$ is the subset $G \cap (C \times B)$.

Special types of functions

Definition. Given a set A, the ***identity function***, denoted by $\mathbf{id}_A$ or $\mathbf{1}_A : A \to A$, is the function whose graph is the set of all (x, y) such that $y = x$. It is an important special case of the inclusion mapping described above with $C = A$.

Identity maps and composition of functions satisfy the following simple but important condition.

Proposition. *If* $f : A \to B$ *is a function, then we have* $\mathbf{1}_B \circ f = f = f \circ \mathbf{1}_A$.

Proof. Let $x \in A$ be arbitrary. Then we have $\mathbf{1}_B \circ f(x) = \mathbf{1}_B(f(x)) = f(x)$ and we also have $f(x) = f(\mathbf{1}_A(x)) = f \circ \mathbf{1}_A(x)$. We can now apply proposition on equality of functions to conclude that the three functions $\mathbf{1}_B \circ f$, f, and $f \circ \mathbf{1}_A$ are equal.■

One familiar example of an identity function is given on (a subset of) the real line by the familiar formula $f(x) = x$. Another example the identity permutation on a set of n letters where n is some positive integer.

More defintions. Let $f : A \to B$ be a function.

1. The function f is ***one – to – one*** or $1-1$ if for all $x, y \in A$, we have $f(x) = f(y)$ if and only if $x = y$. Such a map is also said to be ***injective*** or an ***injection*** or a ***monomorphism*** or an ***embedding*** (sometimes also spelled ***imbedding***).
2. The function f is ***onto*** if for every $y \in B$ there is some $x \in A$ such that $f(x) = y$. Such a map is also said to be ***surjective*** or a ***surjection*** or an ***epimorphism***.
3. The function f is ***$1-1$ and onto*** (or ***$1-1$ onto*** or a ***$1-1$ correspondence***) if it is both $1-1$ and onto. Such a map is also said to be ***bijective*** or a ***bijection*** or an ***isomorphism***. If $A = B$ is a finite set, such a map is often called a ***permutation*** of A.

The following observation is a direct consequence of the definitions.

Proposition. *Let $f:A \to B$ be a function. Then f is surjective if and only if its range is equal to its codomain, or equivalently if and only if $f\,[A] = B$.*■

This follows immediately because the range of f is equal to $f\,[A]$ by definition.

Examples of injections. If A is a set and C is a subset of A, then the previously defined inclusion mapping $i:C \to A$ is an injection because $i(x) = x$ for all x, so that the condition $i(x) = i(y)$ is equivalent to saying that $x = y$. On the other hand, the inclusion i is a surjection if and only if $C = A$.

Examples of surjections. Let A and B be sets, and let $A \times B$ denote their Cartesian product. The (coordinate) projection mappings $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ onto A and B respectively are defined by $\pi_A(x, y) = x$ and $\pi_B(x, y) = y$. These are also called the projections onto the first $(A-)$ and second $(B-)$ coordinates. If both A and B are nonempty, then these mappings are always surjective. On the other hand, the projection π_A is injective if and only if B consists of a single point, and likewise the projection π_B is injective if and only if A consists of a single point.

Logical independence of injectivity and surjectivity. The standard way of showing independence is to give an example of a function that is injective but not surjective and an example that is surjective but not injective. For the former, consider the elementary function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \arctan x$. This is defined for all real numbers and is strictly increasing, so it is automatically ***injective, but it is not surjective*** because its range is the open interval $(-\pi/2, \pi/2)$. An example of a function that is ***surjective but not injective*** is given by $f(x) = x^3 - x$. The function is surjective because for each y one can find a real solution to the cubic equation $x^3 - x = y$. However, it is not injective because $f(0) = f(1) = f(-1) = 0$.■

Observe also that the function $f(x) = x^2$ is ***neither injective nor surjective*** because $f(1) = f(-1)$ and it is not possible to find a real number x such that $x^2 = -1$.■

The following simple factorization principle turns out to be extremely useful for many purposes:

Proposition (Injective – surjective factorization). *Let $f:A \to B$ be a function. Then f is equal to a composite $j \circ q$, where $q: A \to C$ is surjective and $j : C \to B$ is injective.*

Proof. Let **C** be the image of f, and define q such that the graphs of q and f are equal. Take j to be the inclusion of C in B (hence it is injective). By construction q is surjective, and it follows immediately that $f(x) = j(q(x))$ for all x in A.■

Notes. The factorization of a function into a surjection followed by an injection is rarely unique, but there is a close relationship between any two such factorizations whose proof is left to the exercises for this section. Another exercise proves the existence of a second factorization $Q \circ J$, where Q is surjective and J is injective. This also turns out to be useful in certain contexts.

Proposition (Composition and injections/surjections). *Let $f: A \to B$ and $g: B \to C$ be functions.*

(1) *If f and g are surjections, then so is $g \circ f$.*

(2) *If f and g are injections, then so is $g \circ f$.*

(3) *If f and g are bijections, then so is $g \circ f$.*

Proof. The third statement follows from the first two, so it suffices to prove these assertions.

***Verification of* (1):** Assume f and g are onto. Let $c \in C$ be arbitrary. Since g is onto we can find some $b \in B$ such that $g(b) = c$. Since **f** is also onto there is some $a \in A$ such that $f(a) = b$. But then $g \circ f(a) = g\big(f(a)\big) = g(b) = c$. Therefore $g \circ f$ is onto.

***Verification of* (2):** Assume f and g are $1-1$. Take elements $a_1, a_2 \in A$ and suppose that $g \circ f(a_1) = g \circ f(a_2)$. Then $g\big(f(a_1)\big) = g\big(f(a_2)\big)$ by the definition of a composite. Therefore $f(a_1) = f(a_2)$ because g is $1-1$; since f is also $1-1$ it follows next that $a_1 = a_2$. This shows that $g \circ f$ is $1-1$.■

If a function $f: A \to B$ is either $1-1$ or onto, then one can prove strengthened forms for some of the results on images and inverse images of subsets with respect to f.

Theorem. *If $f: A \to B$ is a function, then the image and inverse image constructions for f have the following properties:*

1. *If f is $1-1$ and C is a subset of A, then $C = f^{-1}[f[C]]$.*
2. *If f is onto and D is a subset of B, then $f\big[f^{-1}[D]\big] = D$.*

Proof. As in the proof of the earlier result, we treat each statement separately.

***Verification of* (1):** By the earlier result, we already know $C \subset f^{-1}[f[C]]$. Suppose now that f is $1-1$ and $y \in f^{-1}[f[C]]$. By definition we know that $f(y) = f(x)$ for some $x \in C$. Since f is $1-1$ this implies $y = x$, and therefore we must have $y \in C$. Hence the two sets under consideration are equal if f is $1-1$.

***Verification of* (2):** By the earlier result, we already know $f\left[f^{-1}[D]\right] \subset D$. Suppose now that f is onto, and let $y \in D$. Then there is some $x \in f^{-1}[D]$ such that $y = f(x)$. Therefore y must belong to $f\left[f^{-1}[D]\right]$ if f is onto, proving containment in the other direction in this case.■

Inverse functions

In some situations, it is possible to undo the results of a function by taking the ***inverse*** function. For example, the cube root function is the inverse of x^3, the natural logarithm function is the inverse of e^x, and $\arctan x$ is the inverse to $\tan x$ if the latter is viewed as a function which is defined on the open interval $(-\pi/2, \pi/2)$. Frequently we say that a function is ***invertible*** if an inverse exists. It turns out that a function is only invertible if it is a bijection.

Definition. Let $f: A \rightarrow B$ be a function. A function $g: B \rightarrow A$ is said to be an ***inverse function*** to f if for all $a \in A$ we have $g(f(a)) = a$ and for all $b \in B$ we have $f(g(b)) = b$. By the definition of the identity function, this is equivalent to the conditions $g \circ f = 1_A$ and $f \circ g = 1_B$.

Elementary examples. If A denotes the real numbers, B denotes the positive real numbers, and $f(x) = e^x$, then f has an inverse function g which is the logarithm of x to the base e. Similarly, if $A = B = \mathbb{R}$ and $f(x) = 2x + 4$, then f has an inverse g given by $g(x) = \frac{1}{2}x - 2$. Clearly many other examples of this sort arise in trigonometry and calculus.

Proposition (Characterization of inverse functions). *Let $f: A \rightarrow B$ be a bijection, and define a **function** $f^{-1}: A \rightarrow B$ by taking $f^{-1}(b)$ to be the unique $a \in A$ such that $f(a) = b$; equivalently, the graph of f^{-1} is the set of all ordered pairs (y, x) such that (x, y) lies in the graph of f. Then f^{-1} is well-defined, and it is an inverse of f (in fact, it is the **unique** inverse in view of the next proposition).*

The condition on the graph can be restated as "$x = f^{-1}(y)$ *if and only if* $y = f(x)$."

PROOF. There is at least one a such that $b = f(a)$ since f is onto, and there cannot be more than one such a since f is $1-1$. Therefore f^{-1} is a well – defined function. Since the graph of f^{-1} is the set of all ordered pairs $(f(x), x)$ the definitions imply that f^{-1} satisfies the conditions for being an inverse to f.■

Proposition (Functions with inverses are bijections). *Let $f : A \to B$ be a function. If f has an inverse g, then f is a bijection and the inverse is unique (and it is equal to f^{-1} as defined above).*

Proof. Assume that the mapping f has an inverse g. To show that f is onto, take $b \in B$. Then $f(g(b)) = b$, so b lies in the image of f. To show that f is $1-1$, consider an arbitrary pair of elements $a_1, a_2 \in A$, and suppose that $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$, and since $g \circ f = 1_A$ it follows that $a_1 = a_2$. To show that the inverse is unique, suppose that g and h are both inverses to f. We must show that $g = h$. Let $b \in B$ be arbitrary. Then $f(g(b)) = f(h(b)) = b$ because g and h both inverses, and since f is $1-1$ we must have $g(b) = h(b)$ for all b. By the proposition on equality of functions, we conclude that $g = h$.■

In view of the preceding proposition, one way of showing that a function is a bijection is to show that it has an inverse.

The construction sending a bijective function to its inverse has several basic properties that are summarized in the next result.

Proposition (Identities involving inverse functions). *The inverse function construction has the following properties:*

1. *Let A be a set. Then the identity map 1_A is a bijection, and it is equal to its own inverse.*
2. *Suppose that $f : A \to B$ and $g : B \to C$ are bijections so that their composite $g \circ f$ is also a bijection by a previous result. Then the function $(g \circ f)^{-1}$ is equal to $f^{-1} \circ g^{-1}$.*
3. *If $f : A \to B$ is a bijection with inverse f^{-1}, then $f^{-1} : A \to B$ is also a bijection, and its inverse is equal to f.*

Proof. We shall derive all of these from the conditions $v \circ u = \mathbf{id}_X$ and $u \circ v = \mathbf{id}_Y$ which characterize a function $u : X \to Y$ and its inverse $v : Y \to X$. If $u = \mathbf{id}_A$ then we also have $v = \mathbf{id}_A$ because $\mathbf{id}_A \circ \mathbf{id}_A = \mathbf{id}_A$, proving the first part. To prove the second part, we take $X = A$, $Y = C$, and $u = g \circ f$. If we now set v equal to $f^{-1} \circ g^{-1}$, then the propostion on the associativity property for compositions and the proposition on composites with identity maps combine to imply that the composites $v \circ u$ and $u \circ v$ are both identity maps. Finally, if we take $X = B$, $Y = A$, and $u = f^{-1}$, then $v = f$ has the property that the composites $v \circ u$ and $u \circ v$ are both identity maps.■

Example. Here is an illustration of the identity $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ using the $1-1$ and onto functions $f : \mathbb{R} \longrightarrow (0, \infty)$ defined by $f(x) = e^x$ and $g : (0, \infty) \longrightarrow (0, 1)$ defined by $g(y) = y/(1+y)$ as examples: The composite function $g \circ f$ is given by $z = e^x/(1+e^x)$, and if we solve this for z we obtain the equation

$$x = \ln(z/(1-z)).$$

Since $g^{-1}(z)$ is equal to the expression inside the parentheses and $\ln y = x$ is the inverse to $y = e^x$, this example does satisfy the formula for finding the inverse function of a composite.■

Comment (caution). If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a $1-1$ onto function given in terms of the functions studied in a first year calculus course, ***there is no guarantee that its inverse function can also be expressed in such terms.*** One relatively simple example is the Lambert $W-$ function $W(z)$ which is given by the identity

$$W(z)\, e^{W(z)} = z.$$

Additional details are given in the following subdirectory file:

http://math.ucr.edu/~res/math144-2022/week04/lambert-fcn.pdf