Functions and equivalence relations

Finally, here are some results related to Section 3.5 of Cunningham. We begin with the most basic relationship between the two concepts mentioned in the subheading.

<u>Equivalence class projection.</u> Let A be a set and let $\mathcal{E}$ be an equivalence relation on A; as before, denote the set of equivalence classes for $\mathcal{E}$ by the quotient $A/\mathcal{E}$.. Then the ***<u>equivalence class projection</u>*** (or ***<u>quotient</u>***) map $h_{\mathcal{E}}: A \to A/\mathcal{E}$ is the map which sends a \in A to its equivalence class [a] with respect to $\mathcal{E}$.

<u>Theorem (Passage to quotients).</u> *Let $f: A \to B$ be a function, let $\mathcal{E}$ be an equivalence relation on A, let $h_{\mathcal{E}}: A \to A/\mathcal{E}$ be the quotient map for the equivalence relation as above, and assume that $a_1\, \mathcal{E}\, a_2$* implies $f(a_1) = f(a_2)$ for all $a_1, a_2 \in A$. *Then there is a unique function $f^*: A/\mathcal{E} \to B$* such that $f = f^* \circ h_{\mathcal{E}}$.

$$\begin{tikzcd} A \arrow[r, "h_{\mathcal{E}}"] \arrow[dr, "f"'] & A/\mathcal{E} \arrow[d, "f^*"] \\ & B \end{tikzcd}$$

<u>PROOF.</u> We want to define f^* by the formula $f^*([x]) = f(x)$. What could go wrong? We need to exclude the possibility that there might be $x, y \in A$ such that $[x] = [y]$ but $f(x) \neq f(y)$. In other words, we want to verify that $[x] = [y]$ implies $f(x) = f(y)$. But the latter follows from the assumption that $a_1\, \mathcal{E}\, a_2$ implies $f(a_1) = f(a_2)$. ■

Two special cases of this result are particularly worth mentioning. The second one is also shown in Section 3.5 of Cunningham.

<u>Corollary 1.</u> *Let $f: A \to B$ be a function, let $\mathcal{E}$ and $\mathfrak{D}$ be equivalence relations on A and B respectively, let $h_{\mathcal{E}}: A \to A/\mathcal{E}$ and $h_{\mathfrak{D}}: B \to B/\mathfrak{D}$ be the quotient maps for the equivalence relations as above, and assume that $a_1\, \mathcal{E}\, a_2$ implies $h_{\mathfrak{D}} \circ f(a_1) = h_{\mathfrak{D}} \circ f(a_2)$. Then there is a unique map $f^{**}: A/\mathcal{E} \to B/\mathfrak{D}$* such that $h_{\mathfrak{D}} \circ f = f^* \circ h_{\mathcal{E}}$.

This follows if we replace the map f in the theorem by the composite $h_{\mathfrak{D}} \circ f$.■

Corollary 2. *Let* $f : A \to B$ *be a function, let* $\mathcal{E}$ *be an equivalence relation on* A*, llet* $h_{\mathcal{E}} : A \to A/\mathcal{E}$ *be the quotient map for the equivalence relation as above, and assume that* $a_1 \,\mathcal{E}\, a_2$ implies $f(a_1)\ \mathcal{E}\ f(a_2)$ for all $a_1, a_2 \in A$. *Then there is a unique function* $f^{**} : A/\mathcal{E} \to A/\mathcal{E}$ such that $h_{\mathcal{E}} \circ f = f^{**} \circ h_{\mathcal{E}}$.

This follows from the preceding corollary if we take $A = B$ and $\mathfrak{D} = \mathcal{E}$.■