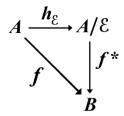
## Functions and equivalence relations

Finally, here are some results related to Section 3.5 of Cunningham. We begin with the most basic relationship between the two concepts mentioned in the subheading.

<u>Equivalence class projection</u>. Let A be a set and let  $\mathcal{E}$  be an equivalence relation on A; as before, denote the set of equivalence classes for  $\mathcal{E}$  by the quotient  $A/\mathcal{E}$ . Then the <u>equivalence class projection</u> (or <u>quotient</u>) map  $h_{\mathcal{E}}: A \to A/\mathcal{E}$  is the map which sends a \in A to its equivalence class [a] with respect to  $\mathcal{E}$ .

<u>Theorem (Passage to quotients)</u>. Let  $f: A \to B$  be a function, let  $\mathcal{E}$  be an equivalence relation on A, let  $h_{\mathcal{E}}: A \to A/\mathcal{E}$  be the quotient map for the equivalence relation as above, and assume that  $a_1 \mathcal{E} a_2$  implies  $f(a_1) = f(a_2)$  for all  $a_1, a_2 \in A$ . Then there is a unique function  $f^*: A/\mathcal{E} \to B$  such that  $f = f^* \circ h_{\mathcal{E}}$ .



**PROOF.** We want to define  $f^*$  by the formula  $f^*([x]) = f(x)$ . What could go wrong? We need to exclude the possibility that there might be  $x, y \in A$  such that [x] = [y] but  $f(x) \neq f(y)$ . In other words, we want to verify that [x] = [y] implies f(x) = f(y). But the latter follows from the assumption that  $a_1 \& a_2$  implies  $f(a_1) = f(a_2)$ .

Two special cases of this result are particularly worth mentioning. The second one is also shown in Section 3.5 of Cunningham.

**Corollary 1.** Let  $f: A \to B$  be a function, let  $\mathcal{E}$  and  $\mathcal{D}$  be equivalence relations on A and B respectively, let  $h_{\mathcal{E}}: A \to A/\mathcal{E}$  and  $h_{\mathcal{D}}: B \to B/\mathcal{D}$  be the quotient maps for the equivalence relations as above, and assume that  $a_1 \mathcal{E} a_2$  implies  $h_{\mathcal{D}} \circ f(a_1) = h_{\mathcal{D}} \circ f(a_2)$ . Then there is a unique map  $f^{**}: A/\mathcal{E} \to B/\mathcal{D}$  such that  $h_{\mathcal{D}} \circ f = f^{*} \circ h_{\mathcal{E}}$ .

This follows if we replace the map f in the theorem by the composite  $h_{\mathcal{D}} \circ f$ .

**Corollary 2.** Let  $f: A \to B$  be a function, let  $\mathcal{E}$  be an equivalence relation on A, llet  $h_{\mathcal{E}}: A \to A / \mathcal{E}$  be the quotient map for the equivalence relation as above, and assume that  $a_1 \mathcal{E} a_2$  implies  $f(a_1) \mathcal{E} f(a_2)$  for all  $a_1, a_2 \in A$ . Then there is a unique function  $f^{**}: A / \mathcal{E} \to A / \mathcal{E}$  such that  $h_{\mathcal{E}} \circ f = f^{**} \circ h_{\mathcal{E}}$ .

This follows from the preceding corollary if we take A = B and  $\mathfrak{D} = \mathfrak{E}.\blacksquare$