SOLUTIONS FOR WEEK 04 EXERCISES

Cunningham, Exercises 3.3

1. DIREGARD THIS ONE.■

2. Let $F : X \to Y$ be the function with $A \subset B \subset X$, and suppose that $y = F(x)$ for some $x \in A$. Then $x \in A$ implies $x \in B$, and therefore $y = F(x)$ implies that $y \in F[B]$.■

7. Let $x \in F^{-1}[C]$, so that $F(x) \in C$. Since $C \subset D$ we also have $F(x) \in D$. By definition this means that $x \in F^{-1}[D]$.■

8. Suppose that $y \in C$. Since F is onto there is some x such that $F(x) = y$, and by definition of the inverse image we have $x \in F^{-1}[C]$. Since $F^{-1}[C] \subset F^{-1}[D]$, we also know that $x \in F^{-1}[D]$, and it follows that $y = f(x) \in D$ also holds. Therefore we have $C \subset D$.■

11. By the definition of a function in the Cunningham, $f \subset g$ should be interpreted as "the graph of f is contained in the graph of g," and similarly if f and g are interchanged. Write a function in $\mathcal{C}$ as $f : A_f \to B_f$.

To show that $\bigcup \mathcal{C}$ is the graph of a function, we need to verify that if x is the first coordinate of an element in $\mathcal{C}$ then there is exactly one ordered pair of the form (x, y) in $\bigcup \mathcal{C}$. By the assumption on x we know that at least one such point exists, so it suffices to show that there cannot be two such points. But suppose that f and g have graphs C_f and C_g with $(x, y) \in C_f$ and $(x, y') \in C_g$. One of these graphs contains the other; without loss of generality we might as well assume $C_f \subset C_g$ (for the other case, switch the roles of f and g in the argument which follows). Then we know that $(x, y) \in C_f \subset C_g$. Since $(x, y') \in C_g$ and C_g is also the graph of a function, we must have $y = y'$. Therefore to every $x \in \bigcup_f A_f$ there is a unique $y \in \bigcup_f B_f$ such that $(x, y) \in \bigcup \mathcal{C}$ and hence the latter is the graph of a function.■

15. The condition $G(X) = G(Y)$ for $X, Y \subset A$ translates to $F[X] = F[Y]$. If $x \in X$, then $F(x) \in F[X]$ and $F[X] = F[Y]$ imply that $F(x) = F(y)$ for some $y \in Y$. Since F is 1–1 it follows that $x = y$ and hence $X \subset Y$. If we switch the roles of X and Y in the preceding two sentences we also conclude that $Y \subset X$; combining these, we have $X = Y$.■

Cunningham, Exercises 3.5

1. We shall prove the contrapositive. If the equivalence relation $\mathcal{F}$ contains more than one point, then the quotient mapping $f : A \to A/\sim$ is not 1–1. This is straightforward: If $u, v \in A$ satisfy $u \neq v$ but $[u] = [v]$, then $f(u) = f(v)$, which means that f is not 1–1.■

8. It is helpful to reformulate the problem more precisely. Let $m = p_1^{r_1}...p_x^{r_x}$ be a factorization of m as a product of distinct primes, and let $n = q_1^{s_1}...q_y^{s_y}$ be a factorization of n. Then $m \sim n$ if and only if $\sum r_i = \sum s_j$. By the unique factoriztion of a positive integer as a product of prime numbers, we know that the sums of the exponents do not depend upon the choice of factorization (note we are not assuming the primes are distinct). Define $\sigma(n)$ to be the sum of the exponents $\sum s_j$.

(a) YES. We need to verify that $\sigma(m) = \sigma(n)$ implies that $\sigma(3m) = \sigma(3n)$. But this follows immediately from the fact that $\sigma(3k) = \sigma(k) + 1$.■

(b) NO. In this case we need to determine whether $[m] = [m']$ and $[n] = [n']$ imply $[m + n] = [m' + n']$. The quickest way to do this is by trial and error with examples involving single digit integers. Let $(m, n) = (5, 2)$ and $(m', n') = (5, 3)$. Then $\sigma(m) = \sigma(n) = \sigma(m') = \sigma(n') = 1$ but $\sigma(5 + 2) = \sigma(7) = 1$ and $\sigma(5 + 3) = \sigma(8) = 3$. Thus the value of $\sigma(m + n)$ depends upon the choices of m and n and not only on their equivalence classes. This means there cannot be a well-defined function h with the desired properties.■

(c) YES. In this case we have $\sigma(mn) = \sigma(m) + \sigma(n)$ because $m = p_1^{r_1}...p_x^{r_x}$ and $n = q_1^{s_1}...q_y^{s_y}$ imply $\sigma(m) = \sum r_i$ and $\sigma(n) = \sum s_j$, so that

$$mn \quad = \quad p_1^{r_1}...p_x^{r_x} \cdot q_1^{s_1}...q_y^{s_y}$$

and $\sigma(mn) = \sum r_i + \sum s_j$.■

The remaining exercises in `exercises04.pdf`

1. (a) This is not a graph for two reasons. First, if $|x| < 1$ then $\left(x, \pm\sqrt{1 - x^2}\right)$ satisfy the equation. Second, if $|x| > 1$ then there are no real values of y such that (x, y) satisfies the equation.■

(b) This is a graph, for if $x \in \mathbb{R}$, then the set consists of all ordered pairs of the form $(x, 1 - x^2)$.■

(c) This is not a graph because there are no ordered pairs of the form $(0, y)$ which satisfy the equation. However, if we replace the first factor of the product by $\mathbb{R} - \{0\}$ then the set **IS** the graph of the function $y = 1/x$.■

(d) This is not a graph because there are infinitely many ordered pairs of the form $(0, y)$ which satisfy the equation; in fact, this is true for all ordered pairs of the form $(0, y)$. However, if we replace the first factor of the product by $\mathbb{R} - \{0\}$ then the set **IS** the graph of the function $y = 0$.■

2. Strictly speaking we should consider $P_1 \times P_2$ where P_1 is all past presidents and P_2 consists of all presidents; whatever hopes one has for the future, there is no way of defining an ordered pair (*current president*, *successor*). The set is not a graph because it contains the two ordered pairs (Cleveland, B. Harrison) and (Cleveland, McKinley).■

3. The set $A \times \{x\}$, which corresponds to the constant function $f(a) = x$ for all a, clearly satisfies the conditions to be a graph. There cannot be any other functions, for if we have $g : A \to \{x\}$ then the only possibility for $g(a)$ is x. Therefore there is one and only one function $A \to \{x\}$.■

4. If $f : \emptyset \to X$ is a function, the only possible choice for the graph $G \subset \emptyset \times X = \emptyset$ is the empty set, and the condition for this to be a graph is vacuously true because there is no a such that $a \in \emptyset$.

If there were a function $h : X \to \emptyset$ with X nonempty, then its graph G would be a subset of $X \times \emptyset = \emptyset$. Since there is no way of finding $y \in \emptyset$, then there are no pairs of the form $(x, y) \in G \subset X \times \emptyset = \emptyset$. In particular, if $a \in X$ there are no ordered pairs of the form $(a, y) \in G$. Since there are no possible graphs, there cannot be any functions $X \to \emptyset$.■

5. The image is $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. It might be easier to look at the function $g(n)$ consisting of all nonzero digits which do appear in the decimal expansion of n. There is at least one nonzero digit, and hence the number which do not appear is at most equal to 8, and clearly there are values of n where this value is realized (take 10 to some positive integral power). Given $2 \leq k \leq 9$, consider the positive integer whose decimal expansion is the first k positive integers, say taken in order. This yields a positive integer for which $9 - k$ of the digits do not appear. If k runs through the integers $2, ..., 9$, then $9 - k$ runs through the integers $0, ..., 7$. Combining these observations we conclude that the image is equal to $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$.■

6. It will be convenient to use the following properties of a linear function like f defined on a closed interval, which are often worked out in precalculus courses:

(1) The maximum and minimum values are given by the values of the function at the endpoints.

(2) Every value y between the minimum and maximum is $f(x)$ for some x in the closed interval.

These imply that we can read off the answers once we know the maximum and minimum values over the intervals in the problem.

(a) We need to solve the equations $3a - 7 = -7$ and $3b - 7 = 2$. The respective solutions are $a = 0$ and $b = 3$, and therefore the inverse image is equal to $[0, 3$.■

(b) We need to solve the equations $y = f(x) = 3x - 7$ when $x = 2$ and $x = 6$. Since $f(2) = -1$ and $f(6) = 11$, the image is equal to $[-1, 11]$.■

7. (a) $f[A] = \{2, 3, 5\}$.■

(b) If $S = \{1, 2, 3\}$ then $f[S] = \{3, 5\}$.■

(c) If $S = \{1, 2, 3\}$ then $f^{-1}[S] = \{1, 4, 5\}$.■

8. If $f(x) = 2x + 1$ and $g(x) = x^2 - 2$, then

$$f \circ g(x) \quad = \quad 2(x^2 - 2) \ + \ 1 \quad = \quad 2x^2 \ - \ 3$$

and similarly

$$g\circ f(x) \quad = \quad (2x+1)^2 - 2 \quad = \quad 4x^2 \ + \ 4x \ - \ 1. \ \blacksquare$$

9. We shall get the most insight into this problem by finding a formula for the inverse function and seeing when it is not definable. In other words, we need to solve

$$y \quad = \quad \frac{x-2}{x-3}$$

for y in terms of x and inspect the result.

If we multiply both sides by the denominator $x-3$ we obtain

$$y(x-3) \quad = \quad (x-2) \qquad \text{or equivalently} \quad yx - 3y \quad = \quad x-2\ .$$

The equation on the right is equivalent to $xy - x = 3y - 2$, which in turn is equivalent to

$$x \quad = \quad \frac{3y-2}{y-1}$$

provided $y \neq 1$. So the inverse function is defined when $y \neq 1$, and if we replace x by $f^{-1}(y)$ in the display we obtain a formula for f^{-1}.■

10. (b) Suppose that $g\circ f$ is 1–1. If $f(x) = f(y)$ then

$$g\circ f(x) \quad = \quad g\,(f(x)) \quad = \quad g\,(f(y)) \quad = \quad g\circ f(y)$$

and since $g\circ f$ is 1–1 this implies $x = y$. Therefore f must also be 1–1.■

(b) Suppose that $g\circ f$ is onto. If $z \in C$, then $z = g\circ f(x) = g\,(f(x))$ for some $x \in A$. Therefore if $y = f(x)$, then $z = g(y)$ and hence g is also onto.■

11. Suppose we are given $C, C' \subset A$ and $D, D' \subset B$ such that $C \times D = C' \times D'$. Then $C = \pi_A[C \times D] = \pi_A[C' \times D'] = C'$ and $D = \pi_B[C \times D] = \pi_D[C' \times D'] = D'$. This implies that h is 1–1.

The image of h consists of "rectangular" open subsets which are products of subsets of A with subsets of B; one property of such a subset Q is that $(x, v),\ (u, y) \in Q$ implies $(x, y) \in Q$. Clearly there are non-rectangular subsets for many (most) choices of A and B, but we need an explicit example. If we take $A = B = \mathbb{R}$ then the set E defined by $xy = 0$, which is the union of the $x-$ and $y-$ axes, is not in the image of h. In fact, the property in the first sentence of this paragraph fails because $(0, 1) \in E$ and $(1, 0) \in E$ but $(1, 1)$ is not in E.■

12. (a) YES. In fact, the map in this example is 1–1 onto. One quick way of seeing this is to notice that the mapping φ in the problem is equal to its own inverse; in other words, $\varphi\circ\varphi$ is the identity.■

(b) NO. The mapping is not 1–1 because a and b are both sent to b, and it is not onto because a does not lie in its image.■

13. (*a*) This map is 1–1 onto, and its inverse is given by solving $y = -3x + 4$ for x in terms of y as follows:

$$y \quad = \quad -3x + 4 \qquad \Longleftrightarrow \qquad \frac{y-4}{-3} \quad = \quad x \;\blacksquare$$

(*b*) This map is not 1–1 because $f(1) = f(-1)$. It is also not onto because $f(x) \leq 7$ (observe that $-3x^2 \leq 0$).

(*c*) This map is not defined if $x = -2$, so it cannot be a 1–1 and onto functions from the set of real numbers to itself.$\blacksquare$

(*d*) This map is 1–1 onto, and its inverse is given by the unique real fifth root of $x^5 - 1$.$\blacksquare$

14. (*a*) We know that $f[A \cap B] \subset f[A] \cap f[B]$ for an aritrary function, so we need only show the reverse inclusion holds if f is 1–1. By definition, $y \in f[A] \cap f[B]$ implies $f(a) = y = f(b)$ for some $a \in A$ and $b \in B$. If f is 1–1 then $a = b$. Since $b \in B$ we must also have $a \in A \cap B$, so that $y = f(a) \in f[A \cap B]$.

Conversely, suppose that $f[A \cap B] = f[A] \cap f[B]$ for all subsets A and B of X. Let $A = \{a\}$ and $B = \{b\}$ where $a, b \in X$ and $a \neq b$. Then we have

$$f[A \cap B] \quad = \quad f[\emptyset] \quad = \quad \emptyset \quad = \quad f[A] \cap f[B]$$

and the latter translates to the statement $f(a) \neq f(b)$. Since a and b were arbitrary, it follows that f is 1–1.$\blacksquare$

(*b*) Suppose that $f[X - A] \subset Y - f[A]$ for all subsets A of X. If X consists of a single point, then it is automatically 1–1, so assume now that X contains elements $a \neq b$ and let $A = \{a\}$, so that $b \in X - A$. Then we have $f(b) \in Y - f[A]$, and since $f(a) \in f[A]$ it follows that $f(a) \neq f(b)$. Therefore f is 1–1.

Conversely, suppose that f is 1–1, and let $A \subset X$. If $y \in f[X-A]$ then y cannot belong to $f[A]$, for $y = f(x)$ for $x \in X - A$ and $y = f(a)$ for $a \in A$ would imply $f(a) = f(x)$, contradicting our assumption that f is 1–1. Since $f(x) \notin f[A]$, it follows that $y = f(x)$ must belong to $Y - f[A]$, and since y was an arbitrary element of $f[X - A]$ the latter set must be contained in $Y - f[A]$.$\blacksquare$

(*c*) Suppose that $Y - f[A] \subset f[X - A]$ for all subsets A of X. If $A = X$ then this relation yields $Y - f[A] \subset f[X - A] = \emptyset$, which implies that $Y = f[X]$. Therefore f is onto.

Conversely, suppose that f is 1–1, and let $A \subset X$ be the empty set. Then $f[A] = \emptyset$ and therefore the inclusion relationship becomes $Y \subset f[X]$. Since the reverse implication is immediate from the definition of $f[X]$, we have $Y = f[X]$ and hence f is onto.$\blacksquare$

16. We want to solve the equation $y = x/(1 + |x|)$ for x in terms of y. This is a little awkward because the absolute value of x is part of the given equation, and one way to deal with this is to consider the cases $x \geq 0$ and $x < 0$ separately.

Suppose first that $x \geq 0$, so that $y = x/(1+x)$ and the range of the function is contained in the nonnegative real numbers. In this case the equation is equivalent to $y(1+x) = x$, which in turn is equivalent to $y + yx = x$ or $y = x(1-y)$, so that

$$x \quad = \quad \frac{y}{1-y} \quad = \quad \frac{y}{1-|y|}$$

where the last equation holds because $y \geq 0$ when $x \geq 0$.

Suppose now that $x \leq 0$ so that $y = x/(1-x)$ and the range of the function is contained in the nonpositive real numbers. In this case the equation is equivalent to $y(1-x) = x$, which in turn is equivalent to $y - yx = x$ or $y = x(1+y)$, so that

$$x \quad = \quad \frac{y}{1+y} \quad = \quad \frac{y}{1-|y|}$$

where the last equation holds because $y \leq 0$ when $x\, eq0$. Observe that we obtained the same formula for x regardless of whether $y \geq 0$ or $y \leq 0$.■

17. Given $h : C \to A \times B$, let $f = \pi_A \circ h$ and $g = \pi_B \circ h$. It then follows that $h(x) = (\, f(x),\ g(x)\,)$. Conversely, if h is given by the preceding formula, then it satisfies $f = \pi_A \circ h$ and $g = \pi_B \circ h$. Therefore the consruction sending h to (g, f) is onto. Conversely, if we start with (f, g) and form h, then its coordinate projections are f and g; therefore if (f, g) and (f', g') yield the same function h, then $f = f'$ and $g = g'$.■

18. We need to show that $T_{B,A} \circ T_{A,B}$ is the identity on $A \times B$ and $T_{A,B} \circ T_{B,A}$ is the identity on $B \times A$:

$$T_{B,A} \circ T_{A,B}(x,y) \quad = \quad T_{B,A}(y,x) \quad - \quad (x,y)$$

$$T_{A,B} \circ T_{B,A}(y,x) \quad = \quad T_{A,B}(x,y) \quad - \quad (y,x)$$

These chains of equations imply that the two functions in questions are inverse to each other.■

19. (a) We shall first prove that $\pi_B \circ j$ is 1–1 onto. In fact, this composite is the identity map on B because $\pi_B \circ j(b) = \pi_B(a,b) = b$. To see that j is 1–1, note that $j(b) = j(b')$ implies that $b = \pi_B \circ j(b) = \pi_B \circ j(b') = b'$.■

(b) First observe that $\varphi(x,y) = j \circ \pi_B(x,y) = j(y) = (a,y)$. This means that $\varphi \circ \varphi(x,y) = \varphi(a,y) = (a,y)$, which yields the identity $\varphi \circ \varphi = \varphi$.■

20. Let $Q : A \times B \to B$ be coordinate projection, and let $J : A \to A \times B$ send $a \in A$ to $(a,\ f(a))$. If $x \in A$, then $Q \circ J(a) = Q\,(a,\ f(a)) = f(a)$, so $f = Q \circ J$. The map Q is onto because it is a coordinate projection and both A and B are nonempty. The map J is 1–1 because $J(a) = J(a')$ implies $a = \pi_A\,(a,\ f(a)) = \pi_A\,(a',\ f(a')) = a'$.■