Math 144
Winter 2022

SOLVED PROBLEMS FOR WEEK 04

1. Let A, B and Y be sets, let $X = A \cup B$, and let $f : A \to Y$ and $g : B \to Y$ be functions such that $f|(A \cap B) = g|(A \cap B)$. Prove that there is a unique function $h : X \to Y$ such that $h|A = f$ and $h|B = g$.

SOLUTION.

Let $\Gamma(f)$ and $\Gamma(g)$ be the graphs of f and g respectively, and let $\Gamma(h) = \Gamma(f) \cup \Gamma(g)$. Suppose that $x \in A \cup B$. If $x \in A - B$ then $(x, f(x))$ is the unique element of $\Gamma(h)$ whose first coordinate is x, and likewise if $x \in B - A$. By construction the first coordinate of an element in $\Gamma(h)$ lies in $X = A \cup B$; we only need to examine what happens if $x \in A \cap B$. In this case we know that if $(x, f(x)) \in \Gamma(h)$ and $(x, g(x)) \in \Gamma(h)$, and if $(x, y) \in \Gamma(h)$ it must be one of these ordered pairs. By assumption $x \in A \cap B$ implies that $f(x) = g(x)$, so if $x \in A \cap B$ we again conclude that there is exactly one ordered pair in $\Gamma(h)$ whose first coordinate is x. This completes the proof that $\Gamma(h)$ is the graph of a function from X to Y. This function is unique, for if k is another such function and $x \in X$ then $x \in A \cup B$, and we have $h(x) = f(x) = k(x)$ if $x \in A$ and $h(x) = f(x) = k(x)$ if $x \in B$. Since the functions have the same values at every element of X we have $h = k$.■

2. Let $f : A \to B$ be a function. Explain why $f^{-1}[B - f[A]]$ is the empty set.

SOLUTION.

If $x \in f^{-1}[B - f[A]]$ then $f(x) \in B - f[A]$. On the other hand, by definition we know that $f(x)$ must belong to $f[A]$, which yields a contradiction. The source of the contradiction is assuming that the inverse image is nonempty, so this assumption must be false and hence the inverse image must be empty.■

3. (a) Suppose A is a set and $E \subset C \subset A$. Prove that the associated inclusion maps satisfy the following identity:

$$i_{E \subset A} = i_{C \subset A} \circ i_{E \subset C}$$

(b) In the setting above, let $f : A \to B$ be a function. Prove that the restriction mappings satisfy the identity $(f|C)|E = f|E$.

SOLUTION.

(a) Let $x \in E$. Then by the definition of inclusion mappings we have

$$i_{C \subset A} \circ i_{E \subset C}(x) = i_{C \subset A}(i_{E \subset C}(x)) = i_{C \subset A}(x) = x$$

and similarly $i_{E \subset A}(x) = x$. Since both functions have the same value at each $xx \in E$, they are equal.■

(b) By definition $f|E = f \circ i_{E\subset A}$. If we now apply the first part and use associativity of composition we may rewrite the right hand side as

$$f \circ \left(i_{C\subset A} \circ i_{E\subset C}\right) \quad = \quad \left(f \circ i_{C\subset A}\right) \circ i_{E\subset C} \quad = \quad (f|C) \circ i_{E\subset C} \quad = \quad (f|C)\,|E$$

which is what we wanted to prove.■

4. Let

$$f(x) \quad = \quad \frac{2x-3}{5x-7} \quad \left(x \;\neq \frac{7}{5}\right) \, .$$

It turns out that f defines a 1–1 onto mapping from $\mathbb{R} - \{\frac{7}{5}\}$ to $\mathbb{R} - \{\frac{2}{5}\}$. Find an explicit formula for f^{-1}.

SOLUTION.

We need to solve the equation

$$y \quad = \quad \frac{2x-3}{5x-7} \quad \left(x \;\neq \frac{7}{5}\right)$$

for x in terms of y. If we begin by multiplying both sides by the denominator $5x - 7$ we obtain $y(5x - 7) = 2x - 3$. Rearranging terms we obtain the equation $5xy - 2x = 7y - 3$, or equivalently $x(5y - 2) = 7y - 3$. The latter yields the desired formula for the inverse function:

$$x \quad = \quad \frac{7y-3}{5y-2} \quad \left(y \;\neq \frac{2}{5}\right) \blacksquare$$

5. Suppose we have a function $f : A \to B$ and two factorizations of f as $j_0 \circ q_0$ and $j_1 \circ q_1$ where the maps q_t are onto and the maps j_t are 1–1 for $t = 0, 1$. Denote the codomain of q_t (equivalently, the domain of j_t) by C_t. Then there is a unique bijection $H : C_0 \to C_1$ such that $H \circ q_0 = q_1$ and $j_1 \circ H = j_0$.

SOLUTION.

Given $y \in C_0$ we would like to define H by choosing $x \in A$ so that $q_0(x) = y$ [such an x exists because q_0 is onto] and setting $H(y) = q_1(x)$.

In order to make such a definition it is necessary to show that the construction does not depend upon the choice of x; in other words, if $q_0(z) = q_0(x) = y$, then $q_1(x) = q_1(z)$. All we have at our disposal are the injectivity and surjectivity assumptions along with the identities $f = j_0 \circ q_0 = j_1 \circ q_1$. Since j_1 is injective, if we can show that $j_1 \circ q_1(z) = j_0 \circ q_0(x)$, then we will have $q_1(z) = q_1(x)$ as desired. But

$$j_1 \circ q_1(z) \quad = \quad f(z) \quad = \quad j_0 \circ q_0(z) \quad = \quad j_0(y) \quad = \quad j_0 \circ q_0(x) \quad = \quad f(x) \quad = \quad j_1 \circ q_1(x)$$

so we do have the necessary identity $q_1(z) = q_1(x)$. Therefore we have defined a mapping $H : C_0 \to C_1$ such that $H \circ q_0 = q_1$.

We now need to show that H is bijective. Suppose that $H(y) = H(y')$ and choose x, x' so that $y = q_0(x)$ and $y' = q_0(x')$. We then have

$$j_0(y) \quad = \quad j_0 \circ q_0(x) \quad = \quad f(x) \quad = \quad j_1 \circ q_1(x) \quad = \quad j_1 H(y)$$

$$j_0(y') \quad = \quad j_0 \circ q_0(x') \quad = \quad f(x') \quad = \quad j_1 \circ q_1(x') \quad = \quad j_1 H(y')$$

and since $H(y) = H(y')$ it follows that the expressions on both lines are equal, so that $j_0(y) = j_0(y')$. Since j_0 is injective, this implies $y = y'$ and hence H is injective. To show that H is surjective, express a typical element $z \in C_1$ as $q_1(x)$ for some x; then if $y = q_0(x)$ we have $z = H(y)$. This completes the proof that H is bijective.

All that remains is to show that H is unique. Suppose that $K : C_0 \to C_1$ also satisfies $K \circ q_0 = q_1$. Then if $y \in C_0$ and $y = q_0(x)$ we have

$$K(y) \quad = \quad K \circ q_0(x) \quad = \quad q_1(x) \quad = \quad H \circ q_0(x) \quad = \quad H(y)$$

and hence $K = H$, proving uniqueness.■