Math 144
Winter 2022

EXERCISES FOR WEEK 05

The exercises in this document deal with partial orderings. Exercises on material from Lecture 10 will be posted next week.

0. Work the following problems from Cunningham:

Exercises 3.4 (pp. 68–69): 4 – 5, 7 – 10, 13, 15

1. Let J be the set of closed intervals in the real numbers, and define a binary relation $\mathcal{B}$ such that $[a, b]\ \mathcal{B}\ [c, d]$ if and only if either $[a, b] = [c, d]$ or $b < c$.

(a) Show that $\mathcal{B}$ defines a partial ordering on J.

(b) Show that two elements of J are comparable with respect to $\mathcal{B}$ (one of them precedes the other) if and only if they are equal or disjoint.

(c) Show that $\mathcal{B}$ is not a linear ordering on J.

2. Let S be the set $\{1, , n\}$. Prove that $\mathcal{P}(S)$ contains a linearly ordered subset T with $n + 1$ elements but $\mathcal{P}(S)$ does not contain a linearly ordered subset with $n + 2$ elements.

3. Let $\mathbb{R}[t]$ be the set of all polynomials with real coefficients, and define $p \leq q$ if and only if $p(x) \leq q(x)$ for all real values of x. Prove this defines a partial ordering of $\mathbb{R}[t]$, but that this partial ordering is not a linear ordering.

4. Let A and B be partially ordered sets, and suppose that $f : A \to B$ is a function such that if $x < y$ in A then $f(x) < f(y)$ in B.

(a) Prove that f is 1–1 if A is linearly ordered.

(b) Give an example where f is not 1–1 and A is not linearly ordered.

5. Suppose that we are given a partially ordered set $A = \{a, b, c, d, e, f, g, h\}$ where the ordering is completely specified by the following relations:

a precedes both b and c.

b precedes both e and g.

c precedes both d and g.

e precedes f.

Both d and g precede h.

Expand the given partial ordering to a linear ordering.