L09

Partial and linear orderings

In many areas of mathematics it is important to compare two objects of the same type and determine whether one is less than or equal to the other. The real number system is one obvious example of this sort, but it is not the only one. When we consider the family of all subsets of a given set, it is often important to know if one subset is contained in another. In both cases the associated ordering by size can be expressed in terms of a binary relation $\mho$, and if $\mho$ is one of these examples it has the following three properties:

$\mho$ is ***reflexive***: $a \mho a$ for all $a \in A$.

$\mho$ is ***antisymmetric***: $a \mho b$ and $b \mho a$ imply $a = b$ for all $a, b \in A$.

$\mho$ is ***transitive***: $a \mho b$ and $b \mho c$ imply $a \mho c$ for all $a, b, c \in A$.

These examples and properties lead to a general concept.

__Definition.__ If A is a set, then a ***partial ordering*** on A is a binary relation $\mho$ on A which is reflexive, antisymmetric and transitive. A ***partially ordered set*** (or ***poset***) is an ordered pair $(A, \mho)$ consisting of a set A together with a partial ordering $\mho$ on A. If the partial ordering $\mho$ is clear or unambiguous from the context, we shall often write $x \mho y$ in a more familiar form like $x \leq y$ or $y \geq x$. Similarly, if $x \leq y$ or $y \geq x$ but $x \neq y$ then we often write $x < y$ or $y < x$ and say either that x ***is strictly less than*** y or equivalently that y ***is strictly greater than*** x.

__ALGEBRAIC EXAMPLE 1.__ Let A be the positive integers and let $\mho$ be the binary relation $x \mho y$ if and only if y is evenly divisible by x (with no remainder; in other words, $y = xz$ for some positive integer z). This relation is reflexive because $x = x \cdot 1$. To see the relation is antisymmetric, suppose that $y = xz$ and $x = yw$. Combining these, we obtain the equation $x = xzw$, where x, z and w are all positive integers. The only way one can have an equation of this sort over the positive integers is if $z = w = 1$. To see that the relation is transitive, suppose we have $y = xu$ and $z = yv$. Combining these, we see that $z = yuv$, where both u and v are positive integers. This implies that $x \mho z$.

__ALGEBRAIC EXAMPLE 2.__ Take A to be the chessboard (checkerboard?) set

$$A = \{1, 2, 3, 4, 5, 6, 7, 8\} \times \{1, 2, 3, 4, 5, 6, 7, 8\}$$

and start with the standard ordering on the first eight positive integers. One then has the so – called ***lexicographic*** or ***dictionary ordering*** on A with $(x, y) \leq (x', y')$ if and only if either (i) $x < x'$, or else (ii) $x = x'$ and $y \leq y'$. We shall show this is a partial ordering by proving a more general statement.

Proposition (Lexicographic ordering on a product). *Suppose that P and Q are partially ordered sets (with orderings denoted by $\leq_P$ and $\leq_Q$), and define a new binary relation $\leq$ (the **lexicographic** or **dictionary ordering**) on the product $P \times Q$ by $(x, y) \leq (x', y')$ if and only if either (i) $x < x'$, or else (ii) $x = x'$ and $y \leq y'$. Then the relation $\leq$ defines a partial ordering on $P \times Q$.*

PROOF. We being by showing the binary relation is reflexive. By Condition (ii) we have $(x, y) \leq (x, y)$.

Suppose now that we have both $(x, y) \leq (x', y')$ and $(x', y') \leq (x, y)$. Then (i) and (ii) combine to show that $x \leq_P x'$ and $x' \leq_P x$; therefore we must have $x = x'$. We can now apply (ii) to conclude that $y \leq_Q y'$ and $y' \leq_Q y$, so that $y = y'$. Thus both coordinates of (x, y) and (x', y') are equal, and consequently the two ordered pairs are equal.

Finally, suppose that we have $(x, y) \leq (z, w)$ and also $(z, w) \leq (u, v)$. The remaining argument splits into cases; as noted before, by definition of the lexicographic relation, if two ordered pairs (a, b) and (c, d) are related then $a \leq c$. ***Case 1***: Suppose we have either $x <_P z$ or $z <_P u$. In either case we have $x <_P u$ and therefore by Condition (1) we have $(x, y) \leq (u, v)$. ***Case 2***: Suppose that $x = z = u$. In this case Condition (2) implies $y \leq_Q w$ and $w \leq_Q v$, so by transitivity of $\leq$ it follows that $y \leq_Q v$. Combining the statements in the last two sentences, we conclude that $(x, y) \leq (u, v)$. This completes the proof of transitivity.■

Linear orderings

One major difference between the ordering of the real numbers and the ordering of a set of subsets is that real numbers satisfy the following ***trichotomy principle***:

For every x and y, exactly one of $x = y$, $x < y$ or $y < x$ is true.

It is easy to construct examples showing this fails for nearly every set of subsets $\mathcal{P}(A)$. Specifically, if $A = \{1, 2\}$ with $x = \{1\}$ and $y = \{2\}$, then x and y are distinct but neither is a subset of the other.

We can formalize this concept using another definition.

Definition. Let $(A, \mathcal{O})$ be a partially ordered set. Then $\mathcal{O}$ is said to be a ***linear ordering***, a ***simple ordering*** or a ***total ordering*** if for every pair of elements x and y in A, we either have $x \ \mathcal{O}\, y$ or $y \ \mathcal{O}\, x$. — Since a partial ordering is antisymmetric, both conditions hold if and only if $x = y$.

Here are two simple but useful results on ordered sets.

Proposition (Restrictions of orderings to subsets). *Let $(A, \mathcal{O})$ be a partially ordered set, let B be a subset of A, and define $\mathcal{O}|B$ be the restricted binary relation on B defined by $\mathcal{O} \cap (B \times B)$. Then $\mathcal{O}|B$ is a partial ordering on B. Furthermore, if $\mathcal{O}$ is a linear ordering then so is $\mathcal{O}|B$.*

The key observation in the proof is that if x and y belong to B, then $x \ \mathcal{O}|B \ y$ if and only if $x \ \mathcal{O} \ y$. Details of the argument are left to the reader as an exercise.■

Proposition (Lexicographic orderings associated to linear orderings). *If A and B are linearly ordered sets, then the product $A \times B$ with the lexicographic ordering is also linearly ordered.*

PROOF. Suppose we are given (x, y) and (x', y'). Since A is linearly ordered, exactly one of the statements $x <_A x'$, $x = x'$ or $x >_A x'$ is true. In the first and third cases we have $(x, y) < (x', y')$ and $(x, y) > (x', y')$ respectively.

Suppose now that $x = x'$; since B is linearly ordered, exactly one of $y <_B y'$, $y = y'$ or $y >_B y'$ is true. In these respective cases we have $(x, y) < (x', y')$, $(x, y) = (x', y')$ and $(x, y) > (x', y')$.■

Partially ordered sets arise in many different mathematical contexts, and this wide range of contexts generates a long list of properties that a partially ordered set may or may not satisfy. We shall discuss a few of these together with some examples for which the properties are true and others for which the properties are false.

The following type of partially (in fact, linearly) ordered set plays an important role in the mathematical sciences.

Definition. A partially ordered set A is said to be ***well – ordered*** if every nonempty subset has a minimal element a^* such that $a^* \leq x$ for all $x \in A$.

Algebraic Examples. If A denotes the nonnegative integers and one takes the usual ordering, then A is well – ordered. This property is crucial to a powerful method of proof known as ***finite induction.*** On the other hand, the sets of all integers and all positive ***real*** numbers are not well – ordered, and in fact these sets themselves have no minimal elements. Given an integer n we know that $n-1 < n$, and given a positive real number x we know that $0 < x/2 < x$.

Proposition. *Every well – ordered set is linearly ordered.*

PROOF. Let A be the well – ordered set, and let B be a nonempty subset of A. If B does not have at least two elements then there is nothing to prove, so assume that B does have at least two elements. Suppose that x and y are distinct elements of B, and consider the nonempty subset $\{x, y\}$. By the well – ordering assumption we know this set has a least element. If it is x, then we have $x < y$, and if it is y then we have $y < x$.■

Extending partial orderings

There is an important relationship between the divisibility ordering and the usual ordering of the positive integers; namely, if $m \mid n$ then $m \leq n$. There are many situations in which one wants an answer to the following more general question:

Extension Question. Given a partial ordering $\mathfrak{O}$ on a set X, is there always a linear ordering $\mathfrak{L}$ on X such that $u \ \mathfrak{O} \ v$ implies $u \ \mathfrak{L} \ v$?

Such questions also arise in nonmathematical contexts. For example, one may have a list of courses that are required for a degree, with each course having its own prerequisites. We shall consider such an example in an appendix to this lecture.

The answer to the Extension Question turns out to be yes. We do not yet have the tools to prove this in complete generality, but we can give a method proof in the finite case which can be applied to specific examples.

Theorem (Extending partial orderings to linear orderings). *Let* $(A, \mathfrak{O})$ *be a* **FINITE** *partially ordered set. Then* $\mathfrak{O}$ *is contained in some linear ordering* $\mathfrak{L}$ *such that* $u \ \mathfrak{O} \ v$ *implies* $u \ \mathfrak{L} \ v$.

The proof relies upon the following preliminary result:

Lemma. *Let* $(A, \mathfrak{O})$ *be a nonempty* **FINITE** *partially ordered set. Then* A *has a minimal element.*

PROOF OF THE LEMMA. Suppose that the conclusion is false, and assume that A has exactly n elements. Let $x_1 \in A$ be arbitrary. Since there is no minimal element in A we can find $x_2 \in A$ such that $x_2 < x_1$. We can continue in this fashion to conclude that there are also elements x_j for $j = 1, \ldots, \mathrm{n}+1$ such that $x_1 > \ldots > x_{n+1}$.

But this is impossible because A only has n elements, so we have a contradiction. The source of the contradiction is our assumption that A has no minimal element, and it follows that A must have a minimal element.■

PROOF OF THE THEOREM. Let x_1 be a minimal element of A. Extend the binary relation $\mathfrak{O}$ to $\mathfrak{O}_1$ by adjoining all pairs of the form (x_1, y) where y runs through all the elements of A. It follows that $\mathfrak{O}_1$ is also a partial ordering, but now we have $x_1 \neq y$ for all $y \in A$ with respect to the enriched ordering $\mathfrak{O}_1$ (observe that $y\ \mathfrak{O}_1\ y'$ implies $y\ \mathfrak{O}\ y'$ for all $y, y' \neq x_1$). Assuming A has at least two elements we can then find a minimal element x_2 of $A - \{x_1\}$, and form $\mathfrak{O}_2$ on the latter subset by adjoining all ordered pairs $(x_2,\ y)$ where y runs through all the elements of $A - \{x_1\}$. Then we have $x_1 < x_2 < y$ for all $y \in A - \{x_1, x_2\}$ with respect to the enriched ordering $\mathfrak{O}_2$. We may continue in this manner recursively for each $k \leq n$, expanding the ordering to $\mathfrak{O}_k$ and obtaining a sequence of elements $\{x_j\}$ such that $x_1 < \ldots < x_k < y$ for all $y \in A - \{x_1, \ldots, x_{\mathrm{k=1}}\}$. This process terminates with $k = n$, at which point we shall obtain the linear ordering $x_1 < \ldots < x_n$.■

An example will be posted in Lecture 09A.