Example for extending partial orderings to linear orderings

Let $A = \{\mathbf{0, a, b, c, d, e, f}\}$, and take the partial ordering which is uniquely specified by taking the following ordered pairs to be strict inequalties:

(0, a)	**(0, b)**	**(a, c)**
(b, c)	**(a, d)**	**(b, e)**
(c, f)	**(d, f)**	**(e, f)**

This is not a linear ordering because neither **a** nor **e** precedes the other. According to the procedure described in the theorem for obtaining a linear ordering, the first step is to find a minimal element. Going through the possibilities, we see that **0** is a minimal element; in this case we can check that **0** already precedes everything else, so we do not need to add any relations at this step. However, at the second step we find that both **a** and **b** are minimal elements if **0** is deleted. We need to choose one of these, so let us take **a** and add all ordered pairs **(a, x)** where **x** runs through all the elements except **0.** If we now delete **0** and **a** then **b** is still a minimal element, so now add all ordered pairs **(b, y)** where **y** runs through all the elements except **0** and **a.** Now consider the subset $\{\mathbf{c, d, e, f}\}$. The minimal elements here are **c** and **d,** so first add all ordered pairs **(c, z)** where **z** belongs to $\{\mathbf{d, e, f}\}$. Next note that the ordered pairs **(d, e)** and **(d, f)** are already in the original partial ordering, and follow this by noting that the ordered pair **(e, f)** is also aready in the partial ordering. We are now left with only one element, namely **f**, and clearly there is nothing more to do at this point because we now have an ordering in which everything else precedes **f**. In this case it turns out that the original partial ordering has been extended to the usual alphabetical ordering with **0** preceding all the letters. However, we also could have obtained other linear orderings by the same procedure. For example, if we had chosen **b** at the second step we would have obtained $\mathbf{0 < b < a < c < d < e < f}$.