Math 144
Fall 2022

SOLUTIONS FOR WEEK 05 EXERCISES

Cunningham, Exercises 3.4

4. We know that $b \in S$ is an upper bound because we have assumed that $x \leq b$ for all $x \in S$. It is also the least upper bound; if c is an upper bound for S then $b \in S$ implies $b \leq c$, and this is the property which characterizes a least upper bound.■

5. We are given that g and g' are lower bounds for S so we have $g \leq x$ andn $g' \leq x$ for all $x \in S$. Since they are both greatest lower bounds, we also have $g' \leq g$ (since g is a greatest lower bound) and $g \leq g'$ (since g' is a greatest lower bound). By the antisymmetric property of partial orderings, it follows that $g = g'$.■

7. Since a is a smallest element of S and $a' \in S$, we must have $a \leq a'$. Similarly, since a' is a smallest element of S and $a \in S$, we must have $a' \leq a$. By the antisymmetric property of partial orderings, it follows that $a = a'$.■

8. The set $\mathbb{N}_+$ of positive integers is clearly the least upper bound for $\mathcal{C}$.■

9. If $\mathcal{F}$ is the subfamily called P in the text, then the family does not have an upper bound if we restrict the partial ordering to $\mathcal{F}$ because the no finite set contains every subset in this family.■

10. See Additional Exercise 4 below.■

13. This can be viewed as a dual companion to Exercise 4, and it has a similar proof; the only difference is that the directions of inequalities must be reversed.

We know that $a \in S$ is a lower bound because we have assumed that $x \geq z$ for all $x \in S$. It is also the greatest lower bound; if d is a lower bound for S then $d \in S$ implies $d \geq a$, and this is the property which characterizes a greatest lower bound.■

15. If $x \leq y$ then $x \in P_x$ implies $z \leq x$ and by the transitivity of partial orderings we have$z \leq y$, so $x \leq y$ implies $P_x \subset P_y$. Conversely, the latter implies $x \leq y$by definition.

Two partially ordered sets are said to be isomorphic if there is a 1–1 onto mapping $f : X \to Y$ such that $a \leq b$ is true in X if and only if $f(a) \leq f(b)$ is true in Y (in other words, f is order preserving). In the setting of this exercise let $f : X \to \mathcal{F}$ be the map sending x to P_x. By construction this is onto. To see that it is also 1–1, suppose that $P_x = P_y$. Then we have $P_x \subset P_y \subset P_x$, and by the first paragraph we can conclude that $x \leq y \leq x$, so that $x = y$. Finally, the conclusions of the first paragraph also imply that f is order preserving.■

The remaining exercises in `exercises05.pdf`

1. We shall work the parts in order.

(a) The reflexive property follows from the construction. To prove the symmetric property, suppose that $[a,b]\ \mathcal{B}\ [c,d]$ and $[c,d]\ \mathcal{B}\ [a,b]$. If $[a,b] \neq [c,d]$, then by definition we have $b < c$ and $d < a$. Since $a < b$ and $c < d$, this yields the impossible chain of strict inequalities $a < b < c < d < a < b$. Thus the only logical possibility is for the two intervals to be equal. Finally, suppose we have $[a,b]\ \mathcal{B}\ [c,d]$ and $[c,d]\ \mathcal{B}\ [e,f]$. The conclusion is trivial if either $[a,b] = [c,d]$ or $[c,d] = [e,f]$, so let us assume that neither holds. We then have $b < c < d < e < f$, and this implies $[a,b]\ \mathcal{B}\ [e,f]$.

(b) The statement of the exercise is equivalent to the statement that if $[a,b]$ and $[c,d]$ are comparable but unequal, then they must be disjoint. Suppose the two intervals are unequal. Then if $[a,b]\ \mathcal{B}\ [c,d]$, we have $b < c$ which means that the two intervals are disjoint, while if $[c,d]\ \mathcal{B}\ [a,b]$, we have $d < a$ which also means that the two intervals are disjoint.

(c) Consider the intervals $[0,1]$ and $[0,2]$. These intervals are not equal and not disjoint, so by the preceding part of the exercise they cannot be comparable with respect to the partial ordering $\mathcal{B}$.■

2. If we have a linearly ordered chain $S_1 < \cdots < S_m$ then the number of elements in S_k is at least one more than the number of elements in S_{k-1}. Since S_1 contains at least zero elements, this means that the number of elements in S_m is at least $m-1$. Since S has n elements, this means that $m - 1 \leq n$ or $m < n + 2$. To get a linearly ordered set with $n + 1$ elements, start with $S_1 = \emptyset$, and for $1 < k \leq n + 1$ let $S_k = \{1, \cdots, k - 1\}$.■

3. The relation is reflexive because $p(x) \leq p(x)$ for all x. It is antisymmetric because $p(x) \leq q(x)$ for all x and $q(x) \leq p(x)$ for all x imply $p(x) = q(x)$ for all x, and hence $p = q$. It is transitive because $p(x) \leq q(x)$ for all x and $q(x) \leq r(x)$ for all x imply $p(x) \leq r(x)$ for all x, so that $p \leq r$.

To show this partial ordering is not a linear ordering we need to find two polynomials p and q such that p is not less than or equal to q and vice versa. It will be enough to find p and q together with real numbers a and b such that $p(a) > q(a)$ but $p(b) < q(b)$, for the first implies $p \leq q$ is false and the second implies that $p \geq q$ is false.

Specifically, take p to be the constant polynomial with value 1 and let $q(x) = x$. Then we have $q(1) > p(1)$ but $q(-1) < p(-1)$.■

4. (a) If A is linearly ordered and $x \neq y$ in A, then either $x < y$ or $y < x$. In these respective cases we have $f(x) < f(y)$ and $f(y) < f(x)$. Therefore $f(x) \neq f(y)$ in both of the possible cases, and therefore f must be 1–1.■

(b) The simplest possible example is to take the partial ordering of $A = \{a,b,c\}$ determined by $a \leq c$ and $b \leq c$, and to let $f : A \to \{0,1\}$ be defined by $f(a) = f(b) = 0$ and $f(c) = 1$. By constuction this function is not 1–1, but we also know that $x < y$ in A implies $f(x) < f(y)$ in $\{0,1\}$.■

5. We shall modify the method for proving the extendibility which was given in the notes: Instead of finding a minimal element at each step, we shall look at all the choices for minimal elements which arise at each step.

First of all, $a \leq x$ for all $x \in A$, so there is only one choice at the first step. Next, consider $A - \{a\}$. This set has two minimal elements; namely, b and c. We are free to choose either one of these at the second step and the other one at the third. At the fourth step we find that the minimal elements of $A - \{a, b, c\}$ are e, g and d. At steps 4 through 6 we may choose these in whatever order we want. Finally, at step 7 we are left with the minimal elements f and h, and at steps 7 and 8 we can choose these in either order. This modified procedure gives us a total of 24 possibilities for the linear ordering:

$$
\begin{array}{ll}
a < b < c < d < e < g < f < h & a < b < c < d < e < g < h < f \\
a < b < c < g < e < d < f < h & a < b < c < g < e < d < h < f \\
a < b < c < e < g < d < f < h & a < b < c < e < g < d < h < f \\
a < b < c < g < d < e < f < h & a < b < c < g < d < e < h < f \\
a < b < c < d < g < e < f < h & a < b < c < d < g < e < h < f \\
a < b < c < e < d < g < f < h & a < b < c < e < d < g < h < f \\
a < c < b < d < e < g < f < h & a < c < b < d < e < g < h < f \\
a < c < b < g < e < d < f < h & a < c < b < g < e < d < h < f \\
a < c < b < e < g < d < f < h & a < c < b < e < g < d < h < f \\
a < c < b < g < d < e < f < h & a < c < b < g < d < e < h < f \\
a < c < b < d < g < e < f < h & a < c < b < d < g < e < h < f \\
a < c < b < e < d < g < f < h & a < c < b < e < d < g < h < f
\end{array}
$$

This does not give all the possible extensions to linear orderings. For example, there is also the linear ordering $a < b < e < f < c < d < g < h$. In practical applications there can be many reasons for choosing one or more alternatives. For example, if the letters denote steps in manufacturing something, it might be less time-consuming to choose one or more of the linear orderings instead of the others.∎