Power sets and characteristic functions

Lecture 12 will use the following generalization of a result on functions from one finite set to another. In Lecture 10 we stated the result in passing but did not prove it explicitly, so we shall now state and prove it formally for the sake of completeness.

<u>Proposition (Characteristic functions and power sets).</u> *Let A be a set. Then there is a $1-1$ correspondence from $\mathcal{P}(A)$ to the set of functions* $\mathbf{F}(A, \{0, 1\})$.

<u>PROOF.</u> This was shown in Lecture 10 for finite sets, and general the argument is identical to the proof in that case except for the final sentence on the number of elements in $\mathbf{F}(A, \{0, 1\})$; we shall deal with the cardinality of this function set later.

Given $B \subset A$, define its ***<u>characteristic function</u>*** $\chi_B : A \to \{0, 1\}$ by $\chi_B(a) = 1$ if $a \in B$ and by $\chi_B(a) = 0$ if $a \notin B$. By construction the inverse image of $\{1\}$ is equal to B, and therefore $\chi_B = \chi_C$ implies $B = C$. Therefore the characteristic function map $\mathcal{P}(A) \to \mathbf{F}(A, \{0, 1\})$ is $1-1$. But this mapping is also onto, for if f is any function from A to $\{0, 1\}$ then $f = \chi_B$ where B is the inverse image of $\{1\}$.■