Math 144
Fall 2022

SOLUTIONS FOR WEEK 06 EXERCISES

Cunningham, Exercises 5.1

8. The map is 1–1 by the Fundamental Theorem of Arithmetic, which states that the factorization of a positive integer into a product of primes is unique up to rearrangement of factors. Suppose now that $i \leq m$ and $j \leq n$ and either $i < m$ or $j < n$ (this is stronger than the hypotheses in Cunningham). Then $2^i \leq 2^m$ and $3^j \leq 3^n$, with either $2^i < 2^m$ and $3^j < 3^n$. In both of the latter cases we have $2^i 3^j < 2^m 3^n$.∎

12. Suppose that $A - B$ is finite. Then $A = B \cup (A - B)$ where both B and $A - B$ are finite. It follows that A is also finite, contradicting our assumption that A is infinite. The source of the problem is our assumption that $A - B$ is finite, and therefore $A - B$ must be infinite.∎

16. If $f : \{1, ..., n\} \to A$ is onto, then by the Cross Section Property we know that $|A| \leq |\{1, ..., n\}| = n$.∎

Cunningham, Exercises 5.2

7. Since A and B are countable sets, there are 1–1 maps $f : A \to \mathbb{N}$ and $g : B \to \mathbb{N}$. Define $f \times g : A \times B \to \mathbb{N} \times \mathbb{N}$ by $f \times g(a, b) = (\,f(a), g(b)\,)$. This map is 1–1, for $f \times g(a, b) = f \times g(a', b')$ implies $f(a) = f(a')$ and $g(b) = g(b')$. Since f and g are both 1–1, we have $a = a'$ and $b = b'$, so that $(a, b) = (a', b')$.

By definition we have $|A \times B| \leq |\mathbb{N} \times \mathbb{N}|$, and since $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$ it follows that $|A \times B| \leq |\mathbb{N}|$.∎

14. Follow the hints. **Note:** This exercise uses material from Lecture 13 in Week 07 and it might have been better to place in the `exercises07.pdf` file.

Step 1. — A countable union of countable sets is countable. — PROOF: Let $A_0,\ A_1,\ ...$ be the countable family of countable sets, and suppose that $A_i = \{a_{i,0},\ a_{i,1}\ ...\ \}$. Let $D \subset \mathbb{N} \times \mathbb{N}$ be the set of all (i, j) such that $a_{i,j}$ is defined (note that the collection of sets might be finite or there might be finite sets in the collection, in which case $a_{i,j}$ might not be definable). Let $\alpha : D \to U = \bigcup_i\ A_i$ be the map $\alpha(i, j) = a_{i,j}$. By construction this map is onto.

Since $D \subset \mathbb{N} \times \mathbb{N}$ and the latter is countable, we know that D is countable. By the Cross Section Property we have $|U| \leq |D|$, which means that U must also be countable.∎

Step 2. — The set Δ_n of polynomials of degree n is countable if $n > 0$, and likewise for the set Δ_0 of constant polynomials (there are reasons for and against taking the degree of the zero polynomial to be 0, and this avoids such arguments). — PROOF: The polynomials of degree n are in 1–1 correspondence with a subset of $\mathbb{Z}^{n+1}$, where we define the latter recursively by $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Z}^{k+1} = \mathbb{Z}^k \times \mathbb{Z}$; it turns out that $|\mathbb{Z}^n| = |\mathbb{N}|$ (see `lecture13.pdf` for more on this; one can derive this result in a manner similar to the proof that $|\mathbb{R}^n| = |\mathbb{R}|$). More precisely, if the polynomial is $a_n t^n + ... + a_1 t + a_0$, map it to the coefficient sequence $(a_n, \ ... \ , a_0)$. We then have $|\Delta_n| \leq |\mathbb{Z}^{n+1}| = |\mathbb{N}|$. On the other hand, there are infinitely many choices for the leading term of the polynomial, so we also have $|\mathbb{N}| \leq |\Delta_n|$. We can now apply the Schröder-Bernstein Theorem to conclude that Δ_n is countable.∎

Final Step. We have decomposed the algebra of polynomials $\mathbb{Z}[t]$ into a countable union $\bigcup_n \ \Delta_n$, and in Step 2 we showed that each of the sets Δ_n is countable. We cann now apply Step 1 to conclude that their union, which is the entire polynomial algebra, is also countable.∎

Cunningham, Exercises 5.3

1. Since B is nonempty there is some $b_0 \in B$. If we define $\beta_0 : A \to A \times B$ to be the slice embedding $\beta_0(a) = (a, b_0)$, then β_0 is 1–1. Therefore we have $|A| \leq |A \times B|$. Since we also have $|\mathbb{N}| < |A|$, this implies $|\mathbb{N}| < |A \times B|$.∎

2. Suppose that $A - B$ is countable. We know that $A = (A \cap B) \cup (A - B)$. Since B is countable and $A \cap B \subset B$, it follows that $A \cap B$ is also countable, and therefore the first two sentences imply that A is a union of two countable sets and hence is countable. This contradicts the assumption that A is not countable; the source of the contradiction is our assuming that $A - B$ was countable, so this must be false and $A - B$ must be uncountable.∎

3. This is a special case of the preceding exercise with $A = \mathbb{R}$ and $B = \mathbb{Q}$.∎

4. If B were countable, then $A \subset B$ would imply $|A| \leq |B| \leq |\mathbb{N}|$, so that A would also be countable. Since A is not countable by assumption, it follows that B must also be uncountable.∎

7. For each $b \in B$ let $C_b : A \to B$ be the constant function whose value is always b. Then $x \neq y$ implies that the images of C_x and C_y are nonempty and disjoint, and therefore $C_x \neq C_y$. This means that $|B| \leq |\mathbf{F}(A, B)|$. Since B is uncountable, it follows that $\mathbf{F}(A, B)$ is also uncountable.∎

Cunningham, Exercises 5.4

1. Let $f(x) = 2 + 3x$; we need only verify that f is 1–1 and onto, and we can do this by finding an inverse function. If we solve $y = 2 + 3x$ for x, we obtain the formula

$$f^{-1}(y) \quad = \quad x \quad = \quad \frac{y-2}{3}$$

and direct calculation yields $f^{-1} \circ f(x) = x$ and $f \circ f^{-1}(y) = y$ for all x and y in the respective open intervals. Therefore we have a 1–1 correspondence from the open interval $A = (0,1)$ to the open interval $B = (2,5)$ and consequently we have $|A| = |B|$.∎

12. Previously we showed that the function

$$f(x) \quad = \quad \frac{x}{1+|x|}$$

defines a 1–1 correspondence from $\mathbb{R}$ to the open interval $(-1,1)$. If we compose this with the inclusion $(-1,1) \to [-\pi,\pi)$ we see that $|\mathbb{R}| \leq$ the cardinality of $[-\pi,\pi)$. On the other hand, if we consider the inclusion $[-\pi,\pi) \subset \mathbb{R}$ we obtain the reverse inequality of cardinal numbers. Therefore we can apply the Schröder-Bernstein Theorem to conclude that $[-\pi,\pi)$ and $\mathbb{R}$ have the same cardinal number.∎

16. The hypotheses imply that $|A| \leq |B| \leq |C| = |A|$, so that $|A| \leq |B|$ and vice versa. Apply the Schröder-Bernstein Theorem to conclude that $|A| = |B|$.∎

19. The mapping h is 1–1, for $h(a,b) = h(a',b')$ implies $f(a) = f(a')$ and $g(b) = g(b')$, which in turn imply $a = a'$ and $b = b'$, so that $(a,b) = (a',b')$. Given $(c,d) \in K \times L$ we also know that $c = f(a)$ and $d = g(b)$ for suitable a and b, and all this translates into the equation $(c,d) = h(a,b)$, showing that h is also onto.∎

34. If we can show that $\mathbb{R}$ and the interval $(0,1)$ have the same cardinality, then by the preceding Exercise 19 and $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$ we can conclude that $(0,1)^2$ and $(0,1)$ have the same cardinal number. As noted before, we have constructed a specific 1–1 correspondence from $\mathbb{R}$ to the interval $(-1,1)$, so it is enough to construct a 1–1 correspondence from the latter to the interval $(0,1)$. Once again we can do this by means of a linear function which sends -1 to 0 and 1 to itself (see the first exercise from this section). The explicit formula is $f(x) = \frac{1}{2}(x+1)$.∎

The remaining exercises in `exercises06.pdf`

1. Follow the hint and define C to be all integers of the form $n + b$ where $b \in B$. Then C is a nonempty subset of $\mathbb{N} \subset \mathbb{Z}$ and as such it has a least element c_0. We claim that $b_0 = c_0 - n$ is a (acturally, the) least element of B. By construction we have $c_0 - n \in B$ and $n + b_0 \in C$. Given an arbitrary element $b \in B$ we have $n + b \in C$, and since c_0 is minimal in C it follows that $c_0 = n + b_0 \leq n + b$. Subtracting n from each side we have $b_0 \leq b$ and hence b_0 is a minimal element of B.∎

2. Suppose that $1 \leq k \leq 10$. How many pairs of the form (k,m) lie inin the set? Regardless of whether k is even or odd, there are 5 choices (odd numbers if k is even, and even numbers is k is odd). Now the number of first choices is 10, so the total number of choices is $10 \times 5 = 50$.∎

3. This is similar to the proof for Step 1 in Exercise 5.3.14 from Cunningham given previously. Let $A_0,\ A_1,\ ...$ be the countable family of sets such that $|A_k| = |\mathbb{R}|$ for all k, and let $g_k : A_k \to \mathbb{R}$ be a 1–1 correspondence. We shall denote the union of the A_k by A. Define a mapping $h : \mathbb{N} \times \mathbb{R} \to A$ by $h(n, r) = g_n^{-1}(r)$. This map is onto by construction, and therefore we have $|A| \leq |\mathbb{N} \times \mathbb{R}|$. Since $\mathbb{N} \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R}$ we must have $|\mathbb{N} \times \mathbb{R}| \leq |\mathbb{N} \times \mathbb{R}| = |\mathbb{R}|$ and therefore $|A| \leq |\mathbb{R}|$. Going the other direction, we have $|\mathbb{R}| = |A_0| \leq |A|$. The result now follows from the Schröder-Bernstein Theorem.■

4. Define $f : X \sqcup Y \to X \cup Y$ by the formulas $f(x, 1) = x$ and $f(y, 2) = y$. By construction f is onto, and its restriction to either $X \times \{1\}$ or $Y \times \{2\}$ is also 1–1. Therefore the only way that two elements of $X \sqcup Y$ can go to the same thing in $X \cup Y$ is if an element in $X \times \{1\}$ and an element in $Y \times \{2\}$ have the same image in $X \cup Y$. By definition this can only happen if the image element is in $X \cap Y$. Since the latter is empty, it follows that no such pair of elements can exist. Therefore the mapping f is both 1–1 and onto.■

5. If $b \in f[A]$ choose $g(b) = a$ such that $f(a) = b$. If $b \notin f[A]$ choose $g(b)$ to be some arbitrary element $a_0 \in A$. Then $f \circ g \circ f(a) = f(a')$ where $a' \in A$ is some element such that $f(a') = b$. Therefore we have $f(a) = b = f \circ g \circ f(a)$.■