Math 144
Winter 2022

EXERCISES FOR WEEK 07

0. Work the following problems from Cunningham:

> Exercises 4.3 (pp. 101–102): 9 – 11 [*Notes:* Nonnegative exponents are given recursively by the identities $x^0 = 1$ and $x^{k+1} = x^k\, x$. Prove the stated Laws of Exponents by induction on k. As in `exercises06.pdf`, take ω to be $\mathbb{N}$.]
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> Exercises 5.1 (pp. 115–117): 19
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> Exercises 5.2 (pp. 122–124): 9, 15
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> Exercises 5.4 (pp. 138–141): 28 – 30
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> Exercises 6.1 (pp. 144–145): 6

1. Let $\mathcal{F}(\mathbb{R})$ be the set of all finite subsets of $\mathbb{R}$. Prove that $|\mathcal{F}(\mathbb{R})| = |\mathbb{R}|$. [*Hint:* The solution to one problem from `exercises06.pdf` will be useful.]

2. The number $n!$ (n factorial), which is the number of permutations of the first n positive integers, is defined recursively by $0! = 1$ and $n! = n \cdot (n-1)!$ for all $n > 0$. Prove that $n! \leq n^n$ for all $n > 0$ and strict inequality holds if $n > 1$.

3. Verify that for all positive integers n we have

$$\sum_{k=1}^{n} \frac{1}{k^2+k} \quad = \quad \frac{n}{n+1} \ .$$

4. Let A be a finite set (our "alphabet"). Then the set **String** (A) of *finite strings over* A is given by the union

$$\bigcup_{n=1}^{\infty} A^n \times \{n\}$$

where A^n denotes the n-fold product of A with itself and $\{n\}$ is appended to ensure that the copies of A^m and A^n are disjoint if $m \neq n$. Prove that **String** (A) is countably infinite.

5. Use the Strong Principle of Finite Induction as in `strong-induction1.pdf` to prove that every integer $n > 23$ can be written as a sum $5a + 7b$ for suitable nonnegative integers a and b. Also show that 23 cannot be written in this form.