Math 144
Fall 2022

SOLUTIONS FOR WEEK 07 EXERCISES

Cunningham, Exercises 4.3

Recall our recursive definition: $x^0 = 1$ and $x^{k+1} = x^k\,x$

9. We shall prove $m^n \cdot m^k = m^{n+k}$ by induction on k. If $k = 1$ this is trivial since $m^{k+0} = m = m^k \cdot 1 = m^k \cdot m^0$. Suppose we know that $m^n \cdot m^k = m^{n+k}$ where $k \geq 0$. Then $m^n \cdot m^{k+1} = m^n \cdot m^k \cdot m = m^{n+k} \cdot m$ (the last equation by the induction hypothesis), and the final term is equal to $m^{n+k+1} = m^{n+(k+1)}$ by the recursive definition.■

10. We shall prove $(mn)^k = m^k \cdot n^k$ by induction on k. If $k = 0$ this reduces to $(1\cdot 1)^0 = 1^0 = 1 = 1\cdot 1 = 1^0\cdot 1^0$, and if $k = 1$ this is even more obvious since $x^1 = x$ for all x. Suppose we know that $(mn)^k = m^k \cdot n^k$ for some $k \geq 0$. Then $(mn)^{k+1} = (mn)^k \cdot mn$ by the recursive definition. Applying the previous exercise, we see that this expression is equal to $m^k \cdot n^k \cdot m \cdot n$, and rearranging terms shows the latter is equal to $m^k \cdot m \times n^k \cdot n = m^{k+1} \cdot n^{k+1}$, where the right hand side is obtained from the recursive definition.■

11. We shall prove $(m^n)^k = m^{nk}$ by induction on $k \geq 1$. If $k = 1$ then both the left and right hand sides of the equation reduce to m^n, so the statement is true for $k = 1$ (if $k = 0$ both sides are equal to 1, so this case is also no problem). Assume the equation holds for $k \geq 1$, and consider $(m^n)^{k+1}$.

By the definition of nonnegative integral exponents we have

$$(m^n)^{k+1} \quad = \quad (m^n)^k \cdot m^n$$

and by the induction hypothesis we know that $(m^n)^k = m^{nk}$. Therefore the right hand side of the displayed equation reduces to $m^{nk} \cdot m^n$, and by Exercise 9 above this reduces further to $m^{nk+n} = m^{n(k+1)}$, completing the proof of the inductive step.■

Cunningham, Exercises 5.1

19. Let $V_k = \bigcup_{i \leq k} A_i$. Then $V_1 = A_1$ and hence V_k is finite if $k = 1$. To complete the proof by induction, we want to show that if V_k is finite and $1 \leq k < n$ then so is V_{k+1}. By definition we have $V_{k+1} = V_k \cup A_{k+1}$ where A_{k+1} is assumed to be finite and V_k is finite by the induction hypothesis. These and the formula for the number of elements in a union of two finite sets imply that V_{k+1} is finite, completing the proof of the inductive step. When we reach $k + 1 = n$, we have shown that the whole union is finite.■

Cunningham, Exercises 5.2

9. By assumption there is a 1–1 mapping $f : A \to \mathbb{N}$. Pick some $a_0 \in A$ (we can do this because $A \neq \emptyset$. Define a mapping $g : \mathbb{N} \to A$ as follows: If $a = f(m)$ for some m, then m is unique because f is 1–1, so we can define $g(m) = a$. If m is not in the image of f let $g(m) = a_0$. We then have $g \circ f(a) = a$ for all $a \in A$, so that every $a \in A$ lies in the image of g. Therefore g is onto.∎

15. By Exercise 14, which was assigned in `exercises06.pdf`, we know that the set of all nonconstant polynomials with integral coefficients is countable. Write these polynomials out in a sequence $p_1(t), p_2(t), ...$ and for each k let W_k denote all the (real or complex) real roots of $p_k(t)$. Each of the sets W_k is finite (and the number of elements is at most the degree of P_k), so we have described the set of algebraic numbers as a countable union of finite sets. As in the solution to the previoiusly cited exercise, such a union is countable, and it follows that the set of all algebraic (real or complex) numbers must also be countable.∎

Cunningham, Exercises 5.4

28. By a previous exercise we know that there are 1–2 correspondences from $A \sqcup B$ to $A \cup B$ and from $K \sqcup L$ to $K \cup L$ because $A \cap B = K \cap L = \emptyset$, so it suffices to prove $|A \sqcup B| \leq |K \sqcup L|$. Let $f : A \to K$ and $g : B \to L$ be 1–1 functions. Then one can check directly that $h : A \sqcup B \to K \sqcup L$ defined by $h(a,1) = (\, f(a),\, 1)$ and $h(b,2) = (\, g(b),\, 2)$ is 1–1; by checking the second coordinate one sees that a point in $A \times \{1\}$ and a point in $B \times \{2\}$ cannot go to the same point in $K \sqcup L$, and the restrictions of h to both $A \times \{1\}$ and $B \times \{2\}$ are 1–1 by the choices of f and g. Therefore we have $|A \sqcup B| \leq |K \sqcup L|$.∎

29. Let f and g be as in the previous exercise, and define $h : A \times B \to K \times L$ be the product map $h(a,b) = (\, f(a),\, g(b)\,)$. Then h is 1–1 because $h(a,b) = h(a', b;)$ implies $f(a) = f(a')$ and $g(b) = g(b')$, and these equations imply $a = a'$ and $b = b'$ since f and g are 1–1. Therefore we have $|A \times B| \leq |K \times L|$.∎

30. Once again let f and g be as in the preceding two exercises, but now also let $r : K \to A$ be a mapping such that $r \circ f = 1_A$; to construct such a map let $a_0 \in A$ (the latter is nonempty), and define $r(x) = a$ if $x = f(a)$ (there is only one a such that $f(a) = x$ and $r(x) = a_0$ if $x \notin f[A]$; observe that $r \circ f = 1_A$.

We need to construct a 1–1 mapping from $\mathbf{F}(A,B)$ to $\mathbf{F}(K,L)$. First define $C_1 : \mathbf{F}(K,B) \to \mathbf{F}(K,L)$ sending $h : K \to B$ to the composite $g \circ h$. This construction is 1–1 for $g \circ h = g \circ h'$ implies $g \circ h(x) = g \circ h'(x)$ for all $x \in K$, and since g is 1–1 it follows that $h(x) = h(x')$ for all x, so that $h = h'$. Next define $C_2 : \mathbf{F}(A,B) \to \mathbf{F}(K,B)$ sending $\varphi : A \to B$ to the composite $\varphi \circ r$. This construction is also 1–1, for $\varphi \circ r = \varphi' \circ r$ impliest

$$\varphi \;=\; \varphi \circ 1_A \;=\; \varphi \circ r \circ f \;=\; \varphi' \circ r \circ f \;=\; \varphi' \circ 1_A \;=\; \varphi' \;.$$

Since C_1 and C_2 are 1–1, their composite $C_1 \circ C_2 : \mathbf{F}(A,B) \to \mathbf{F}(K,L)$ is also 1–1 and therefore $|\mathbf{F}(A,B)| \leq |\mathbf{F}(K,L)|$.∎

1. By Exercise 3 from `exercises06.pdf` we know that if a countable family of subsets $A_n \subset \mathbb{R}$ satisfies $|A_n| = |\mathbb{R}|$ for all n, then the countable union $\bigcup_n A_n$ also has the same cardinality of $\mathbb{R}$. Split $\mathcal{F}$ into the pairwise disjoint subfamilies $\mathcal{F}_n$ of subsets with n elements, where n runs over all elements of $\mathbb{N}$. Then $\mathcal{F}_0 = \{\emptyset\}$, and we claim that $|\mathcal{F}_n| = |\mathbb{R}|$ for all $n > 0$. As usual, we shall use the Schröder-Bernstein Theorem.

If $n > 0$, define a 1–1 map $h_n : \mathbb{R} \to \mathcal{F}_n$ sending $a \in \mathbb{R}$ to the set $\{a, a+1, ..., a+n-1\}$, which shows that $|\mathbb{R}| \leq |\mathcal{F}_n|$. Next, define a map $k_n : \mathcal{F}_n \to \mathbb{R}^n$ as follows: Put the n elements of a set $B \in \mathcal{F}_n$ in order (with respect to the usual ordering of $\mathbb{R}$) say $x_1 < ... < x_n$ and let k_n send this set to $(x_1, ...x_n) \in \mathbb{R}^n$. This map is also 1–1, so we have $|\mathcal{F}_n| \leq |\mathbb{R}^n| = |\mathbb{R}|$. We can now apply the Schröder-Bernstein Theorem to conclude that $|\mathcal{F}_n| = |\mathbb{R}|$ for each $n > 0$ and therefore the set $\mathcal{F}_+ = \bigcup_{n>0} \mathcal{F}_n$ also has the same cardinality as $\mathbb{R}$ by the previously cited Exercise 3.

By consstruction we have $\mathcal{F} = \mathcal{F}_+ \cup \mathcal{F}_0$, so we have one more thing to complete. Define a 1–1 mapping from $\mathcal{F}$ to $\mathbb{R}^2$ sending $\mathcal{F}_+$ to the line $\mathbb{R} \times \{0\}$ and sending $\mathcal{F}_0$ to $\{(0,1)\}$. This yields the chain of inequalities $|\mathbb{R}| = |\mathcal{F}_+| \leq |\mathcal{F}| \leq |\mathbb{R}^2| = |\mathbb{R}|$. By the Schröder-Bernstein Theorem this implies that $|\mathcal{F}| = |\mathbb{R}|$.■

2. Since it is debatable whether 0^0 can actually be defined, the problem should have been formulated for $n \geq 1$. In any case we know that $1! = 1 = 1^1$ so the statement is true for $n = 1$. To complete the argument, we need to show that $k! \leq k^k$ implies $(k+1)! < (k+1)^{k+1}$ for all $k \geq 1$.

The inductive step follows from the chain of inequalities

$$(k+1)! \quad = \quad k! \cdot (k+1) \quad \leq \quad k^k \cdot (k+1) \quad < \quad (k+1)^k \cdot (k+1) \quad = \quad (k+1)^{k+1} \; .$$

On its surface, the reasoning might not seem to fit perfectly because two separate statements are involved; namely, $k! \leq k^k$ for $k \geq 1$ and $k! < k^k$ for $k \geq 2$. More precisely, the reasoning for this exercise proceeds as follows: $1! = 1 = 1^1 \Longrightarrow 2! < 2^2 \Longrightarrow 2! \leq 2^2 \Longrightarrow 3! < 3^3 \Longrightarrow 3! \leq 3^3 \Longrightarrow 4! < 4^4 ...$ *etc.*■

3. Let S_n be the statement

$$\sum_{k=1}^{n} \frac{1}{k^2 + k} \quad = \quad \frac{n}{n+1} \; .$$

Then one can verify directly that S_1 is true, so to give a proof by mathematical induction we need to show that S_n implies S_{n+1} for all $n \geq 1$. If S_n is true then by the induction hypothesis we have

$$\sum_{k=1}^{n+1} \frac{1}{k^2+k} \quad = \quad \left(\sum_{k=1}^{n} \frac{1}{k^2+k} \right) \; + \; \frac{1}{(n+1)^2 + (n+1)} \quad = \quad \frac{n}{n+1} \; + \; \frac{1}{(n+1)^2 + (n+1)}$$

so the proof reduces to verifying that the right hand side is equal to

$$\frac{n+1}{n+2}\,.$$

Here is the derivation of the identity that we need:

$$\frac{n}{n+1} + \frac{1}{(n+1)^2+(n+1)} = \frac{n(n+2)}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)} =$$

$$\frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}\,.\blacksquare$$

4. Let W_n (words of length n) be the set $A^n \times \{n\}$. Since A is finite the set W_n is also finite. Therefore **String** (A) is a union of the countable family of sets W_n where n runs through the elements of $\mathbb{N}$. Since a countable union of countable sets is countable, it follows that A is countable. To see that the set is infinite, consider the set of words formed using only one letter in A:

$$(a,\ a,\ \ldots\ , a; n) \qquad \text{where} \quad a \in A \quad \text{and} \quad n \in \mathbb{N}_+$$

Since this subset of **String** (A) is countably infinite, it follows that **String** (A) itself must be infinite.■

5. Let S_n be the statement that n can be written in the form $5a + 7b$ for suitable nonnegative integers a and b. In order to prove this using the Strong Principle of Finite Induction, we must first verify it for a few values of $n \geq 24$:

$$S_{24}: \quad 24 = 10+14 = (5\cdot 2)+(7\cdot 2)$$
$$S_{25}: \quad 25 = (5\cdot 5)$$
$$S_{26}: \quad 26 = 5+21 = (5\cdot 1)+(7\cdot 3)$$
$$S_{27}: \quad 27 = 20+7 = (5\cdot 4)+(7\cdot 1)$$
$$S_{28}: \quad 28 = (7\cdot 4)$$

The inductive step is then given as follows:

For $n \geq 29$, S_k true for $24 \leq k \leq n-1 \quad \Rightarrow$ so is S_n.

We can prove this as follows: If $n \geq 29$, then $24 \leq n-5 \leq n-1$. Therefore S_{n-5} is true, so that $n - 5 = 5c + 7b$ for nonnegative integers b and c. Since we may rewrite this as $n = 5(c+1) + 7b$ we see that $n = 5a + 7b$ where both $a = c+1$ and b are nonnegative integers. Therefore S_n is true, and this completes the proof by the Strong Principle of Finite Induction.■

Finally, we need to show that S_{23} is false. Since 23 leaves a remainder of 3 when divided by 5, any expression of the form $23 = 5a + 7b$ must have $b > 0$. The only possibilities are $b = 1, 2, 3$. However, for each of these possibilities we know that $23 - 7b = 16, 9, 2$ is not evenly divisible by 5, and therefore we cannot write $23 = 5a + 7b$ where a and b are nonnegative integers.■