Math 144
Winter 2022

EXERCISES FOR WEEK 08

0. Work the following problems from Cunningham:

Exercises 7.1 (pp. 163–165): 2, 5, 6

Exercises 7.3 (pp. 122–124): 6

1. (*a*) If A and B are well-ordered sets, show that the disjoint union

$$A \sqcup B \quad = \quad A \times \{1\} \ \cup \ B \times \{2\}$$

is well ordered if we take the ordering such that (*i*) it agrees with the usual orderings on $A \times \{1\}$ and $B \times \{2\}$, (*ii*) the elements of $A \times \{1\}$ precede the elements of $B \times \{2\}$.

(*b*) Show that the well-ordered sets $A \sqcup B$ and $B \sqcup A$ are not necessarily equivalent well-orderings; specifically, if $A = \mathbb{N}$ and $B = \{1\}$ then there is no 1–1 onto map from one to the other that is order-preserving. [*Hint:* If such a 1–1 onto order-preserving map exists between two partially ordered sets, explain why one set has a maximal element if and only if the other does and how this applies to the given examples.]

2. A subset C of a partially ordered set A is said to be **cofinal** if for each $a \in A$ there is some $c \in C$ such that $c > a$. Prove that every linearly ordered set has a cofinal well-ordered subset.

3. Prove that a linearly ordered set is well-ordered if and only if the set of strict predecessors of each element is well-ordered.

4. Let S be a set, and let $\mathcal{F} \subset \mathcal{P}(S)$ be a collection of pairwise disjoint subsets. Prove that there is a subset C of S that has exactly one element in common with each subset A in $\mathcal{F}$. [*Hint:* This statement is logically equivalent to the Axiom of Choice.]

5. Find the mistake in the following argument, whose conclusion is obviously false:

Theorem. *There is a finite set of nonnegative integers which is not a proper subset of another finite set of nonnegative integers.*

Proof. Let $\mathcal{B}$ be the set of finite subsets of $\mathbb{N}$, partially ordered by inclusion. Let $\mathcal{C} \subset \mathcal{B}$ be a linearly ordered subfamily, and let

$$A \quad = \quad \bigcup_{C \in \mathcal{C}} C \ .$$

Since A contains each $C \in \mathcal{C}$, we know that A is an upper bound for the sets in $\mathcal{C}$. By Zorn's Lemma, it follows that $\mathcal{B}$ has some maximal element M, and by definition M is a finite subset which is not properly contained in any other finite subset. THIS IS CLEARLY ABSURD, BUT WHERE IS THE MISTAKE?