Proof that Zorn's Lemma implies the Axioms of Choice and Well – Ordering

We have noted that the Axiom of Choice, the Well – Ordering Axiom and Zorn's Lemma are logically equivalent to each other. Most of these are shown in Cunningham, but the derivation of the Axioms of Choice and Well – Ordering from Zorn's Lemma is not. Since the first two statements are logically equivalent assumptions by the results in Cunningham, it suffices to prove that Zorn's Lemma implies the Axiom of Well – Ordering. We are posting a proof of this for the sake of completeness.

Proving that Zorn's Lemma implies the Well – Ordering Axiom. This is a typical example of how Zorn's Lemma is used in mathematics. The idea is to start with a set X and to consider an auxiliary partially ordered set W of well – orderings, with $\alpha \leq \beta$ if and only if α corresponds to an initial portion of β (in other words, all elements less than or equal to some element of β). Then one shows that W satisfies the hypotheses of Zorn's Lemma and hence W has a maximal element γ. The final step is to check that this maximal element is a well – ordering for the entire set X; if not, then one could construct a larger well – ordering which also contains some element of $X - \gamma$ and this element is greater than everything in γ.■