Math 144
Winter 2022

SOLUTIONS FOR WEEK 08 EXERCISES

Cunningham, Exercises 7.1

2. For each $x \in A$ let $x\,\mathcal{R}$ be the set of all $y \in B$ such that $(x, y) \in \mathcal{R}$. The assumtion on $\mathcal{R}$ implies that each set $x\,\mathcal{R}$ is nonempty. If $c : \mathcal{P}(B) - \emptyset \to B$ is a choice function, define $f(x) = c\,(x\,\mathcal{R})$. Then f is a function by construction, and its graph is contained in $\mathcal{R}$.∎

5. (a) The set $\bigcup\, C$ clearly contains every $\mathbf{c} \in C$, and since $\bigcup\, C \in \mathcal{F}$ we know that the latter is partially ordered. We need to show that it is linearly ordered. Let x_1, x_2 be distinct elements of $\bigcup\, C$, where $x_1 \in \mathbf{c}_1 \in C$ and $x_2 \in \mathbf{c}_2 \in C$. Since C is linearly ordered, one of $\mathbf{c}_1,\ \mathbf{c}_2$ contains the other. Without loss of generality we might as well assume that $\mathbf{c}_2$ is the larger subset. Since x_1 and x_2 both belong to the latter, it follows that either $x_1 \leq x_2$ or $x_1 \geq x_2$ and hence $\bigcup\, C$ is linearly ordered. By construction, it is then an upper bound for C in $\mathcal{F}$.∎

(b) The point of (a) was to prove that the linearly ordered subset $C \subset \mathcal{F}$ has an upper bound in $\mathcal{F}$. Therefore Zorn's Lemma implies that $\mathcal{F}$ has a maximal element M, and as such it is not a proper subset of any $A \in \mathcal{F}$.∎

6. (a) We are looking at the partially ordered set M of all $C \subset A$ such that $f|C$ is 1–1. Let L be a linearly ordered subset of M. We claim that $\bigcup\, L \subset M$. If not, then the restriction of f to $\bigcup\, L$ is not 1–1. Suppose that $x_1, x_2 \in \bigcup\, L$ are are distinct elements such that $f(x_1) = f(x_2)$. As in 5(a) let $x_1 \in \mathbf{m}_1 \in L$ and $x_2 \in \mathbf{m}_2 \in L$. Since L is linearly ordered, one of $\mathbf{m}_1,\ \mathbf{m}_2$ contains the other. Without loss of generality we might as well assume that $\mathbf{m}_2$ is the larger subset. Since x_1 and x_2 both belong to the latter, we know that the restriction of f to $\mathbf{m}_2$ is 1–1 and hence $x_1 = x_2$. This contradicts our earlier assumption that $x_1 \neq x_2$. The source of the contradiction is our assumption that f is not 1–1 on $\bigcup\, L$, so this must be false and we are forced to conclude that the latter is linearly ordered. By construction $\bigcup\, L$ is an upper bound for all the sets in L and $L \in M$. By Zorn's Lemma this means that M must have a maximal element.

(b) In the setting of (a), assume further that f is onto. Let $C \in M$ be maximal, but let us assume that $f[C]$ is a proper subset of B. If $b_0 \in B$ does not lie in $f[C]$, let $a_0 \in A$ be such that $b_0 = f(a_0)$; such an element exists because f is onto. Since $b_0 \notin f[C]$ if follows that $a_0 \notin C$. But this means that the restriction of f to the larger subset $C \cup \{a_0\} \neq C$ will be 1–1 since nothing in C maps to b_0 and $f|C$ is 1–1. This contradicts the maximality of C. The source of the contradiction is our assumption that $f|C$ is not onto, so we conclude that the latter must define a 1–1 onto mapping from C to B.∎

6. Let $X \subset A \times B$ be nonempty, and let X_1 be the set of first coordinates for elements of X. Since X is nonempty it follows that X_1 is too. Let α_0 be the least element of X_1. Next, let $X_2(\alpha_0)$ be the set of all $\beta \in B$ such that $(\alpha_0, \beta) \in X$. This set is also nonempty since there is some element of X with first coordinate α_0. Therefore $X_2(\alpha_0) \subset B$ has a least element β_0. By the definition of the lexicographic order we know that (α_0, β_0) must be the least element of X.■

The remaining exercises in `exercises08.pdf`

1. (a) Suppose that X is a nonempty subset of $A \sqcup B$. If either $X \subset A \times \{1\}$ or $X \subset B \times \{2\}$ then X is in 1–1 order=preserving correspondence with a nonempty subset Y of A or B respectively. The hypotheses imply that Y must have a least element ξ, and the corresponding element (ξ, k) of $X \subset A \times \{1\}$ or $X \subset B \times \{2\}$ will then be the least element of X. If X contains elements of both $X \subset A \times \{1\}$ and $X \subset B \times \{2\}$ then the set $X \cap (A \times \{1\})$ is nonempty and hence the set of all $\alpha \in A$ such that $(\alpha, 1) \in X \cap (A \times \{1\})$ will have a minimal element ξ; the corresponding element $(\xi, 1)$ of $X \cap (A \times \{1\})$ will then be a least element of X.■

(b) We shall use the following fact: *If $f : X \to Y$ is a* 1–1 *onto and strictly order-preserving mapping of partially ordered sets and m is a maximal element of X, then $f(m)$ is a maximal element of Y.* Verifying this is an elementary exercise.

The inequivalence of the well-orderings now follows because $A \sqcup B$ has a maximal element $(1, 2)$ but $B \sqcup a$ has no maximal element.■

2. Choose a well-ordering of $\mathcal{P}(A)$ and try to construct a strictly increasing map from $\mathcal{P}(A)$ **with this well-ordering** to A (forget the partial ordering of $\mathcal{P}(A)$ by inclusion). Let's call this well-ordered set W to avoid confusion; the main thing we need to know is that $|A| < |W|$.

Let $A^+ = A \cup \{A\}$ and extend the partial ordering on A to A^+ by making $A \in A^+$ the maximal element; here we are using the fact that no set is a member of itself. We shall define a nondecreasing map $f : W \to A^+$ by transfinite recursion such that f is strictly increasing on $f^{-1}[A]$.

Denote the minimal element of W by 0, and define $f(0)$ by picking a point in A using a choice function. Suppose now that we have defined $f(\beta)$ for all $\beta < \alpha$; we need to define $f(\alpha)$. There are two cases. If there is some $z \in A$ such that $z > f(\beta)$ for all $\beta < \alpha$, define $f(\alpha)$ by choosing such a value of z (again, this requires a choice function). If no such value of z exists, let $f(\alpha) = A$.

Let $B = f\left[f^{-1}[A]\right]$; since P is well-ordered and f is strictly increasing on on $f^{-1}[A]$, it follows that B is a well-ordered subset of A. Thus it will suffice to show that B is cofinal in A. Suppose that $x \in A$; we need to show that there is some $b \in B$ such that $b > x$. Assume this does not hold for some particular choice of x. If this happens then the recursive definition yields a strictly increasing map from W into A, and in fact the image is contained in the set of all elements less than x. Since f is strictly increasing it follows that $|W| \leq |A|$. However, by construction we have $|W| > |A|$, which yields a contradiction. This means that for each $x \in A$ there must be some b such that $b > x$, so that B is a cofinal well-ordered subset.■

3. One direction is straightforward: If a set X is well-ordered then every nonempty subset is well ordered; strictly speaking, this is only true for nonminimal elements (there are no strict predecessors for the least element), but this omission has no further effect on the solution.

Assume now that for every nonminimal element $x \in X$ the set $P(x)$ of all $y \in X$ such that $y < x$ is well-ordered. Let Y be a nonempty subset of X, and let $xx \in Y$. If x is the least element of Y the conclusion of the result is true. Otherwise the set $Y \cap P(x)$ is nonempty, and as such it has a minimal element y_0. We claim that y_0 is also a minimal element of Y. To see this, let $z \in Y$. Then either $z \geq x$ or $z < x$. If $z < x$ then $z \in Y \cap P(x)$, and since y_0 is the least element of the latter subset it follows that $y_0 \leq z$. If $x \leq z$ then $y_0 < x \leq z$ because $y_0 \in P(x)$. Since X is linearly ordered it follows that y_0 is in fact the least element of Y.■

4. Let c be a choice function on $\mathcal{P}(S)$ and consider the set of all objects of the form $c(F)$ where f runs through the members of $\mathcal{F}$. Since the family is pairwise disjoint, it follows that for each F we have $F \cap C = \{c(F)\}$.■

5. The set A does not belong to $\mathcal{B}$ if A is infinite, and this possibility can be realized. Specifically, let $\mathcal{B}$ be the family of sets $\{0, 1, \ \dots \ , n\}$. Then $A = \mathbb{N}$ and therefore is not finite. In fact, if A is infinite then there is no upper bound for the chain $\mathcal{B} = \mathcal{C}$.■