Transfinite cardinal arithmetic

We are now ready to state and derive the basic facts about the arithmetic of cardinal numbers. In several ways this resembles the arithmetic of nonnegative integers, but in others there are significant differences. Some of these simply reflect results obtained thus far, but there are also many other differences. We shall begin with the basic definitions.

Definition. (Addition of cardinal numbers). If A and B are sets with cardinal numbers $|A|$ and $|B|$ respectively, then the *sum* $|A| + |B|$ is equal to $|A \sqcup B|$.

Definition. (Multiplication of cardinal numbers). If A and B are sets with cardinal numbers $|A|$ and $|B|$ respectively, then the *product* $|A| \times |B|$ or $|A| \cdot |B|$ (or sometimes even $|A||B|$) is equal to $|A \times B|$.

Definition. (Exponentiation of cardinal numbers). If A and B are sets with cardinal numbers $|A|$ and $|B|$ respectively, then the *power operation* (or *exponential operation*) $|A|^{|B|}$ is $|\mathsf{F}(B, A)|$, where $\mathsf{F}(B, A)$ is the set of functions from B to A.

In order to justify these definitions we need to verify two things; namely, that (i) these definitions agree with the counting results in previous lectures when A and B are finite sets, and also (ii) that the construction is ***well − defined***; we have defined the operations by choosing specific sets A and B with given cardinal numbers, and we need to make sure that if choose another pair of sets, say C and D, then we obtain the same cardinal numbers. The first point is easy to check; if A and B are finite sets, then the formulas in Lecture 10 show that the numbers of elements in the finite sets $A \sqcup B$, $A \times B$, and $\mathsf{F}(B, A)$ are respectively equal to $|A| + |B|$, $|A| \cdot |B|$ and $\mathsf{F}(B, A)$. The following elementary result disposes of the second issue.

Proposition (Cardinal arithmetic operations are well − defined). *Suppose that we are given sets* A, B, C, D *and we also have* $1-1$ *onto correspondences* $f : A \to C$ *and* $g : B \to D$. *Then there are* $1-1$ *onto correspondences from* $A \sqcup B$, $A \times B$, *and* $\mathsf{F}(B, A)$ *to* $C \sqcup D$, $C \times D$, *and* $\mathsf{F}(D, C)$ *respectively.*

PROOF. Partial generalizations involving inequalities are proved in the exercises for week 08: If we have mappings f and g which are $1-1$ but not necessarily onto, then there are $1-1$ (but not necessarily onto) mappings from $A \sqcup B$, $A \times B$, and $\mathsf{F}(B, A)$ to $C \sqcup D$, $C \times D$, and $\mathsf{F}(D, C)$ respectively. We claim that these maps have inverses if both f and g have inverses.

The main idea for proving the inequalities was to define mappings

$$H: A \sqcup B \to C \sqcup D, \quad J: A \times B \to C \times D, \quad K: \mathsf{F}(B, A) \to \mathsf{F}(D, C)$$

by the following formulas:

$$H(a, 1) = (f(a), 1), \quad H(b, 2) = (g(b), 2)$$

$$J(a, b) = (f(a), g(b))$$

$$[K(\varphi)](c) = f \varphi g^{-1}(c)$$

We can also define mappings in the opposite direction(s)

$$L: C \sqcup D \to A \sqcup B, \quad M: C \times D \to A \times B, \quad N: \mathsf{F}(D, C) \to \mathsf{F}(B, A)$$

by substituting f^{-1}, g^{-1} and g for the variables f, g and g^{-1} into the corresponding definitions of H, J and K respectively. Routine calculations (left to the reader) show that the maps L, M and N are inverses to the corresponding mappings H, J and K.■

Corollary. *If X is a set then the cardinality of $\mathcal{P}(X)$, the set of all subsets of X, is equal to $2^{|X|}$.*

This follows because there is a $1-1$ correspondence between $\mathcal{P}(X)$ and the set of all functions from X to $\{0, 1\}$.■

We are now in a position to verify that the arithmetic of cardinal numbers has many of the same formal properties as the arithmetic of nonegative integers. We shall start with the rules for addition and multiplication.

Theorem (Arithmetic identities for cardinal numbers). *If α, β, and γ are cardinal numbers for some subsets of a sufficiently large set S, then the sum and product operations obey the following algebraic identities:*

(Associative law of addition) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$

(Commutative law of addition) $\alpha + \beta = \beta + \alpha$

(Associative law of multiplication) $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$

(Commutative law of multiplication) $\alpha \cdot \beta = \beta \cdot \alpha$

(Distributive law) $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$

(Equals added to unequals) $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$

(Equals multiplied by unequals) $\alpha \leq \beta \Rightarrow \alpha \cdot \gamma \leq \beta \cdot \gamma$

(Rules for 0 = cardinality of empty set) $\alpha + 0 = \alpha, \quad \alpha \cdot 0 = 0$

(Rule for 1) $\alpha \cdot 1 = \alpha$

These are all straightforward to verify. For example, the commutative laws are just a restatement that there are $1-1$ onto maps from $A \sqcup B$ to $B \sqcup A$ and also from $A \times B$ to $B \times A$. Similarly, the distributive law is an abstract way of saying that there is a $1-1$ onto map from $A \times (B \sqcup C)$ to $(A \times B) \sqcup (A \times C)$. The rules for 0 and 1 reflect the facts that $A \cup \varnothing = A$, $A \times \varnothing = \varnothing$, and the coordinate projection from $A \times \{1\}$ to A sending $(a, 1)$ to a is $1-1$ onto.■

NOTE. In the statements about inequalities, one cannot draw stronger conclusions if there is a strict inequality $\alpha < \beta$. This will follow from the identities in the next theorem, which restates the addition and multiplication rules involving the first transfinite cardinal number $\aleph_0$:

Theorem (Arithmetic identities involving $\aleph_0$). *We have the following identities involving* $\aleph_0$:

(Idempotent laws) $\aleph_0 + \aleph_0 = \aleph_0, \quad \aleph_0 \cdot \aleph_0 = \aleph_0.$

(Extended idempotent laws) $0 < n < \aleph_0 \Rightarrow n \cdot \aleph_0 = \aleph_0,$

$$(\aleph_0)^n = \aleph_0$$

(Absorption law) $k \leq \aleph_0 \leq \alpha \Rightarrow \alpha + k = \alpha$

The Idempotent law was verified earlier, and the extended idempotent laws follow by finite induction and the definition of nonnegative integral exponents (see the exercises). Here is a proof of the absorption law: Choose A such that $|A| = \alpha$. Since A is infinite, it contains a countably infinite subset D; let $C = A - D$. Then we have $\alpha = |C| + \aleph_0 = |C| + \aleph_0 + \aleph_0 = \alpha + \aleph_0$, proving the identity when $k = \aleph_0$. We can retrieve the case $k < \aleph_0$ from the case $k = \aleph_0$, the chain of inequalities

$$\alpha \leq \alpha + k \leq \alpha + \aleph_0 = \alpha$$

and the Schröder – Bernstein Theorem.■

Examples. The preceding theorem implies that $\aleph_0 + 1 = \aleph_0 + 2$, showing that $\alpha < \beta$ does not necessarily imply $\alpha + \gamma < \beta + \gamma$. Similarly, $1 \cdot \aleph_0 = \aleph_0$ and $\aleph_0 \cdot \aleph_0 = \aleph_0$ show that $\alpha < \beta$ does not necessarily imply $\alpha \cdot \gamma < \beta \cdot \gamma$.

Finally, exponentiation for cardinal numbers satisfies the standard identities which hold for nonnegative integers:

Theorem (Laws of exponents). *If* α, β *and* γ *are (finite or transfinite) cardinal numbers, then the following equations hold:*

$$\gamma^{\alpha+\beta} = \gamma^{\alpha} \cdot \gamma^{\beta}$$

$$(\gamma^{\alpha})^{\beta} = \gamma^{\alpha\beta}$$

$$(\beta\gamma)^{\alpha} = \beta^{\alpha} \cdot \gamma^{\alpha}$$

These will follow from some basic results on sets of functions from one set to another:

Theorem (Exponential laws for sets of functions). *If* **A**, **B** *and* **C** *are sets, then one has the following* $1-1$ *correspondences:*

1. $\mathsf{F}(A \sqcup B, C) \to \mathsf{F}(A, C) \times \mathsf{F}(B, C)$
2. $\mathsf{F}(A, \mathsf{F}(B, C)) \to \mathsf{F}(A \times B, C)$
3. $\mathsf{F}(A, B \times C) \to \mathsf{F}(A, B) \times \mathsf{F}(A, C)$

Hints for proving these exponential laws are given in the exercises for this lecture.■

Corollary. *We have* $2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}$, $2^{\aleph_0} = 2^{\aleph_0} + 2^{\aleph_0}$ *and* $2^{\aleph_0} = \aleph_0 \times 2^{\aleph_0}$.

PROOF. By the first law of exponents we have $2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0 + \aleph_0} = 2^{\aleph_0}$, where the final equation is true because $\aleph_0 + \aleph_0 = \aleph_0$. The remaining equalities are consequences of the following chain of inequalities:

$$2^{\aleph_0} \le 2^{\aleph_0} + 2^{\aleph_0} \le \aleph_0 \times 2^{\aleph_0} \le 2^{\aleph_0} \times 2^{\aleph_0} = 2^{\aleph_0}.■$$

This corollary is essentially a refinement of the previously established result that the cardinalities of $\mathbb{R}$ *and* $\mathbb{R}^n = \mathbb{R} \times \ldots \times \mathbb{R}$ (with $n > 1$ factors) are equal (see Lecture 13). However, it also reflects a key property of addition and multiplication for arbitrary transfinite cardinals; namely, the addition and multiplication tables are very simple and are given by idempotent identities $\alpha + \alpha = \alpha \cdot \alpha = \alpha$ and some straightforward consequences of these facts. We shall prove the idempotent identities in the next lecture.■

Transcendental numbers

A real or complex number a is said to be ***algebraic*** if it is a root of some nontrivial polynomial with integer coefficients, and it turns out that this is equivalent to saying that a is a root of some nontrivial polynomial with rational coefficients; if a is not algebraic, then it is said to be ***transcendental***.

It is not clear when mathematicians first considered the concept of a transcendental number, but various historical facts strongly suggest this took place near the middle of the 17th century. Many mathematicians speculated about the existence of such numbers and whether π or e might be transcendental, but the existence of transcendental numbers was first shown rigorously by J. Liouville in the 1840s. His work gave specific examples including the so – called ***Liouville constant***:

$$c = \sum_{j=1}^{\infty} 10^{-j!} = 0.110001000000000000000001000....$$

During the next few decades, proofs that e and π were transcendental finally appeared; these results were due to C. Hermite and F. Lindemann respectively. Many other easily constructed numbers have been shown to be transcendental numbers since the original results of Liouville, but there are still many open questions that are very easy to state but seem unlikely to be answered in the foreseeable future. For example, it is not known whether πe or $\pi + e$ is transcendental (however, we know that at least one of these numbers is transcendental).

Theorem. (Strong existence theorem for real transcendental numbers – Cantor). *The set of transcendental real numbers is nonempty, and its cardinality is equal to* $2^{\aleph_0}$.

PROOF. By definition the set of real numbers $\mathbb{R}$ splits into a union of the disjoint subsets A of algebraic real numbers and T of transcendental real numbers. By one of the previous exercises we know that $|A| = \aleph_0$. Therefore we have a string of equations $2^{\aleph_0} = |\mathbb{R}| = |A| + |T| = \aleph_0 + |T| = |T|$.■