Transfinite cardinal arithmetic III

This lecture describes some additional results and behavior patterns for transfinite cardinal numbers.

Theorem (Absorption law for exponents). *If* α *is a transfinite cardinal number then* $\alpha^{\alpha} = 2^{\alpha}$.

PROOF. Choose A such that $|A| = \alpha$. Then a function from A to itself is completely determined by its graph, which is a subset of $A \times A$. This immediately yields an upper estimate $\alpha^{\alpha} \leq 2^{\alpha\times\alpha} = 2^{\alpha}$ (in Lecture 17A we showed that $\alpha\times\alpha = \alpha$ for every transfinite cardinal number α). On the other hand, by Theorem **5.4.16(3)** in Cunningham, we also know that $2^{\alpha} \leq \alpha^{\alpha}$. Finally, by the Schröder – Bernstein Theorem these inequalities imply that $\alpha^{\alpha} = 2^{\alpha}$.■

Of course, if $n > 2$ then 2^{n} is considerably smaller than n^{n}, so once again we see that the arithmetic of cardinal numbers is much different than ordinary arithmetic with nonnegative integers in some ways. We shall give one more example.

Definition. If X is a set then the ***symmetric group*** on X, denoted by $\Sigma(X)$, is the set of all $1-1$ and onto mappings from X to itself. Note that if X is finite and has n elements then it follows that $\Sigma(X)$ has $n!$ elements. The next result shows that the cardinality of $\Sigma(X)$ also depends only on the cardinality of $\Sigma(X)$ in the transfinite case.

Proposition. *If* $|X| = |Y|$, *then* $|\Sigma(X)| = |\Sigma(Y)|$.

PROOF. Let $\varphi: X \longrightarrow Y$ be a $1-1$ correspondence and consider the "conjugation" map $C(\varphi): \Sigma(X) \longrightarrow \Sigma(Y)$ which sends the $1-1$ onto map $f: X \longrightarrow X$ to the composite $\varphi\circ f\circ\varphi^{-1}$. This mapping is also $1-1$ onto because it is a composite of $1-1$ onto maps. Furthermore, one can show that $C(\varphi)$ is $1-1$ onto by verifying directly that an inverse is given by $C(\varphi^{-1}): \Sigma(Y) \longrightarrow \Sigma(X)$.■

The cardinality of $\Sigma(X)$ is given explicitly by the next result.

Theorem (Cardinality of infinite symmetric groups). *If* X *is infinite and* $|X| = \alpha$, *then* $|\Sigma(X)| = \alpha^{\alpha} = 2^{\alpha}$.

If X is finite and has n elements, so that $\Sigma(X)$ has $n!$ elements, it seems worth noting that $2^n < n! < n^n$ for $n > 2$, and in fact the differences between consecutive terms in this inequality to to infinity as n tends to infinity.

<u>**PROOF.**</u> By definition $\Sigma(X)$ is contained in the function set $\mathrm{F}(X, X)$, and this yields the inequality $|\Sigma(X)| \leq \alpha^{\alpha}$. By the first theorem and the Schröder – Bernstein Theorem, it will suffice to show that $2^{\alpha} \leq |\Sigma(X)|$, and we shall do this by constructing a $1-1$ map from $\mathcal{P}(X)$ to $\Sigma(X)$.

Since $\alpha + \alpha = \alpha$ we can split X into two subsets X_1 and X_2 such that there are $1-1$ onto maps f_1 and f_2 from X to X_1 and X_2. Let B be a subset of X, and define $T_B: X \longrightarrow X$ as follows:

1. If x is in B then T_B interchanges $f_1(x)$ and $f_2(x)$.
2. If x is not in B then T_B sends $f_1(x)$ and $f_2(x)$ to themselves.

One can check directly that $T_B(x) \neq x$ if and only if x lies in $f_1[B] \cup f_2[B]$ and that T_B is $1-1$ onto; in fact, T_B is equal to its own inverse. Furthermore, if B and D are different subsets of X then T_B and T_D are unequal. Therefore the construction sending B to T_B yields the desired $1-1$ map from $\mathcal{P}(X)$ to $\Sigma(X)$, and as noted in the first paragraph this suffices to complete the proof of the theorem.■

Subsequent material from this lecture will not be covered on the second in – class examination.