The Continuum Hypothesis

It is reasonable to ask whether there are other statements which might deserve to be taken as axioms for set theory. One widely known statement of this type is the the ***Continuum Hypothesis***, which emerged very early in the study of set theory.

CONTINUUM HYPOTHESIS. *If A is an infinite subset of the real numbers $\mathbb{R}$, then either there is a $1 - 1$ correspondence between A and the natural numbers $\mathbb{N}$, or else there is a $1 - 1$ correspondence between A and $\mathbb{R}$.*

This question arose naturally in Cantor's work establishing set theory, one motivation being that he did not find any examples of subsets whose cardinal numbers were strictly between those of $\mathbb{N}$ and $\mathbb{R}$.

Since there is a $1 - 1$ correspondence between the real numbers $\mathbb{R}$ and the set $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$, one can reformulate this as the first case of a more sweeping conjecture known as the ***Generalized Continuum Hypothesis***:

GENERALIZED CONTINUUM HYPOTHESIS. *If S is an infinite set and T is a subset of $\mathcal{P}(S)$, then* ***either***

(*i*) *there is a one-to-one correspondence between T and a subset of S,* **or else**

(*ii*) *there is a one-to-one correspondence between T and $\mathcal{P}(S)$.*

In analogy with his results on the Axiom of Choice, the work of Gödel showed that if a contradiction to the axioms for set theory arose if one assumes the Continuum Hypothesis or the Generalized Continuum Hypothesis, then one can also obtain a contradiction without such an extra assumption. On the other hand, the previously mentioned fundamental work of P. M. Cohen shows that one can construct models for set theory such that the Continuum Hypothesis was true for some models and false for others. In fact, one can construct models for which the number of cardinalities between those of $\mathbb{N}$ and $\mathbb{R}$ can vary to some extent; some aspects of this are discussed below.

Because of Cohen's work, most mathematicians are neutral about assuming either the Continuum Hypothesis or its generalization. One reason resembles the case for assuming the Axiom of Choice: Mathematicians would prefer to include as many objects as possible in set theory so long as these objects do not lead to a logical contradiction. Another reason is that there are still no counterexamples to the Continuum Hypothesis or it generalization which arise in situations of independent interest (all the examples are generated within set theory itself).

Cohen's methods show that several other natural questions in set theory are true in some models but false in others. We shall limit our discussion to a related question concerning cardinal numbers:

Suppose that A *and* B *are sets whose power sets satisfy the cardinal number equation* $|\mathcal{P}(A)| = |\mathcal{P}(B)|$. *Does it follow that* $|A| = |B|$?

For finite sets this is a trivial consequence of the fact that the function 2^x is strictly increasing over the real numbers. For infinite sets, there is a curious relation between this question and the Generalized Continuum Hypothesis: *If the latter is true, then the answer to the cardinality question is* **YES.** This follows because for every infinite set A we know that $|\mathcal{P}(A)|$ is the unique first transfinite cardinal number that is strictly larger than $|A|$, and conversely $|A|$ will be the largest cardinal number that is strictly less than $|\mathcal{P}(A)|$.

On the other hand, the condition on cardinal numbers is not strong enough to imply the Generalized Continuum Hypothesis, and one can also construct models of set theory containing sets A and B such that $|A| < |B|$ but $2^{|A|} = 2^{|B|}$. More generally, very precise results on the possible sequences of cardinal numbers that can be written as $2^{|A|}$ for some $|A|$ are given by results of W. B. Easton which build upon Cohen's methods; Easton's result essentially states that a few relatively straightforward necessary conditions on such sequences of cardinal numbers are also sufficient to realize it as the set of cardinalities for power sets.

Possibilities for the cardinality of the real numbers. Since Cohen's results imply that $|\mathbb{R}|$ may or may not be the next cardinal number after $\aleph_0$ depending upon which model for set theory is being considered, one can ask which cardinal numbers are possible values for $|\mathbb{R}|$. Results on this and more general questions of the same type follow from Easton's work.

In order to give more information on this question, we shall need some material from Section 7.1 of Cunningham. A major result from that section states that *the collection of all cardinal numbers (for subsets of some large universal set) is well – ordered*. If the universal set is sufficiently large, this means that the first few cardinal numbers can be written in an increasing sequence as follows (compare Cunningham, p. 213):

$$0, 1, 2, 3, 4, \dots, \aleph_0, \aleph_1, \aleph_2, \dots, \aleph_\omega, \aleph_{\omega+1}, \dots$$

In particular, it turns out that $|\mathbb{R}|$ can be equal to $\aleph_n$ for every positive integer n (in a suitable model for set theory) but it cannot be equal to the cardinal number $\aleph_\omega$ as defined in Cunningham.