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Mathematics 144, Fall 2006, Examination 1

Answer Key

1. [*20 points*] Let A and B be sets (contained in some large set S). Prove the absorption law $A \cup (A \cap B) = A$.

SOLUTION

We first note that $A \subset A \cup (A \cap B) = A$ because $x \in A$ implies $x \in A$ or $x \in A \cap B$ (in fact, the first is always true). Conversely, if $x \in A \cup (A \cap B) = A$ then $x \in A$ or $x \in A \cap B$. In the first case we know that $x \in A$, while in the second we know that $x \in A \cap B$, so that $x \in A$ and $x \in B$. In either case we see that $x \in A$, and therefore we must have $x \in A$, which implies $A \cup (A \cap B) = A$. Since each set is contained in the other, they are equal.

2. [*20 points*] Let A be the set of all positive integers n such that $100 \leq n \leq 999$, and let E be the binary relation on A given by $x\ E\ y$ if and only if the three digits in x are rearrangements of the three digits in y; then E is an equivalence relation, and you may assume this without proving it. Determine all elements of A that are in the equivalence classes of the number n for each of $n = 111$, 122, and 135. [*Hint:* The combined number of integers in the three equivalence classes is equal to 10.]

SOLUTION

The equivalence class of 111 only contains 111, the equivalence class of 122 is given by $\{122,\ 212,\ 221\}$, and the equivalence class of 135 is given by $\{135,\ 351,\ 513,\ 153,\ 315,\ 531\}$.

3. [*20 points*] Suppose that A, B and C are subsets of some set S. Prove that $A \times (B - C) = (A \times B) - (A \times C)$.

SOLUTION

Once again we show that each is contained in the other. If $u \in A \times (B - C)$, then $u = (x, y)$ where $x \in A$ and $y \in B - C$. It follows that $(x, y) \in A \times B$ but $(x, y) \notin A \times C$, so we have $A \times (B - C) \subset (A \times B) - (A \times C)$.

Conversely, if $u \in (A \times B) - (A \times C)$, then $u = (x, y)$ where $x \in A$ and $y \in B$. However, since the pair does not belong to $A \times C$, we either have $x \notin A$ or $y \notin C$. The first does not happen because we already know that $x \in A$, and therefore we must have $y \notin C$, which means that $y \in B - C$. Therefore $y \in B - C$, so that $(x, y) \in A \times (B - C)$. Therefore we also have $(A \times B) - (A \times C) \subset A \times (B - C)$. Since each of the two sets is contained in the other, they are equal.

4. [*25 points*] (*a*) Let X be the binary operation on the set $\mathbf{Z}$ of (signed) integers given by $a\ X\ b$ if and only if $b = a^r$ for some **positive** integer r. Prove that X defines a partial ordering on $\mathbf{Z}$.

(*b*) If $a\ X\ b$, does it follow that $a \leq b$ in the usual ordering of the integers? Either prove this or give an example of an ordered pair (a, b) where the first relation is true but the second is false.

SOLUTION

(*a*) The relation is reflexive because $a = a^1$. To see it is symmetric, suppose we have $b = a^r$ and $a = b^q$. Then by the laws of exponents we have $a = a^{rq}$, which means that either $a = 0$ or $rq = 1$. In the first case it follows that $b = 0^r = 0 = a$, while in the second it follows that $r = q = 1$ so that $b = a$. To see the relation is transitive, note that if $b = a^r$ and $c = b^q$ then $c = a^{qr}$ again by the laws of exponents.

(*b*) The easiest way to see the answer is **NO** is to take $a = -2$ and $b = -8$, so that $b = a^3$ but $b < a$.

5. [*15 points*] Let $\mathbf{R}$ be the real numbers, and let V be the binary relation on $\mathbf{R}$ defined by $x\ V\ y$ if and only if $y^2 - x^2 = 1$. Give an example to show that V is not transitive; in other words, find a, b, c such that $a\ V\ b$ and $b\ V\ c$ are true but $a\ V\ c$ is false.

SOLUTION

Take $(0,1)$ and $(1, \sqrt{2})$. Both lie in the subset of ordered pairs for which the relation is true, but the pair $(0, \sqrt{2})$ does not because $y^2 - x^2 = 2$ in this case.

There are many other possibilities of this type; for example one can take $a = \sqrt{n}$, $b = \sqrt{n+1}$, $c = \sqrt{n+2}$, where n is an arbitrary nonnegative integer.