Exercises for Unit III (Elementary constructions on sets)

III.1 : Boolean operations

(Halmos, §§ 4 – 5**;** Lipschutz, §§ 1.6 – 1.7**)**

Problems for study.

Lipschutz : 1.9, 1.12 **[*misprint:*** In the first sentence, the expression $\mathbf{B} \cap \mathbf{A}$ should be replaced by $\mathbf{B} \cap \mathbf{A}^{\mathbf{c}}$**]**, 1.18 – 1.19, 1.46 – 1.47, 1.48 (*gh*), 1.50, 1.52, 1.54 – 1.55, 11.14

Exercises to work.

1. Give examples of sets **A, B, C** such that each of the three pairwise intersections $\mathbf{A} \cap \mathbf{B}$, $\mathbf{A} \cap \mathbf{C}$, $\mathbf{B} \cap \mathbf{C}$ is nonempty but the total intersection $\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}$ is empty (this illustrates the importance of recognizing the difference between a disjoint collection of sets and a pairwise disjoint collection of sets).

2. Is it possible to find a non – disjoint collection of sets **A, B, C** such that at least one pairwise intersection is empty? Prove this or give a counterexample.

3. Prove that unions and intersections satisfy the pair of identities $\mathbf{A} \cap (\mathbf{B} \cup \mathbf{A}) = \mathbf{A}$ and $\mathbf{A} \cup (\mathbf{B} \cap \mathbf{A}) = \mathbf{A}$ (these are called the ***absorption laws***).

4. (Rosen, Exercise 18, p. 95) Let **A, B, C** be subsets of some fixed set **S.** Prove that

$$(\mathbf{A} - \mathbf{B}) - \mathbf{C} = (\mathbf{A} - \mathbf{C}) - (\mathbf{B} - \mathbf{C}).$$

5. (Rosen, Exercise 21, p. 95) What can one say about the sets **A** and **B** if we know the following?

(a) $\mathbf{A} \cup \mathbf{B} = \mathbf{A}$.
(b) $\mathbf{A} \cap \mathbf{B} = \mathbf{A}$.
(c) $\mathbf{A} - \mathbf{B} = \mathbf{A}$.
(d) $\mathbf{A} \cap \mathbf{B} = \mathbf{B} \cap \mathbf{A}$.
(e) $\mathbf{A} - \mathbf{B} = \mathbf{B} - \mathbf{A}$.

6. Prove or give a counterexample for the following statement: If **A, B, C** are subsets of some set **S** and $\mathbf{A} \cup \mathbf{C} = \mathbf{B} \cup \mathbf{C}$, then $\mathbf{A} = \mathbf{B}$.

7. Prove or give a counterexample for the following statement: If **A** and **B** are subsets of some set **S** and $\mathbf{A} \cap \mathbf{B} = \mathbf{A} \cup \mathbf{B}$, then $\mathbf{A} = \mathbf{B}$.

8. (Rosen, Exercise 22, p. 95) Can one conclude that $A = B$ if **A** and **B** are subsets of some set **S** and satisfy the following? There are actually **three** parts to this problem, corresponding to whether **(1), (2),** or both are known to be true. Give reasons for your answer in all three cases.

(1) $A \cup C = B \cup C$.
(2) $A \cap C = B \cap C$.

9. (Rosen, Exercise 41, p. 116) Let **A, B, C** be subsets of some fixed set **S.** Show that the set $(A - B) - C$ is not necessarily equal to $A - (B - C)$.

10. (Rosen, Exercise 43, p. 116) Let **A, B, C, D** be subsets of some fixed set **S.** Prove or disprove that $(A - B) - (C - D) = (A - C) - (B - D)$.

11. (Rosen, Exercise 14, p. 95) Let **A, B, C** be subsets of some fixed set **S.** Prove that the following hold:

(a) $A \cup B \subset (A \cup B) \cup C$.
(b) $(A \cap B) \cap C \subset A \cap B$.
(c) $(A - B) - C \subset A - C$.
(d) $(A - C) \cap (C - B) = \emptyset$.
(e) $(B - A) \cup (C - A) = (B \cup C) - A$.

12. Suppose that we are given sets **A, B, C, D** such that $A \subset C$ and $B \subset D$. Prove that $A \cup B \subset C \cup D$ and $A \cap B \subset C \cap D$.

13. Suppose that **A** and **B** are subsets of a given set **S.** Prove that $A \subset B$ if and only if $S - B \subset S - A$.

14. (Halmos, p. 16) Let **A, B,** and **C** be subsets of a given set **S.** Prove that one has the mixed associativity (also known as ***modularity***) property

$$(A \cap B) \cup C = A \cap (B \cup C)$$

if and only if $C \subset A$; in particular, the criterion has nothing to do with **B.**

III.2 : Ordered pairs and products

(Halmos, §§ 3, 6; Lipschutz, §§ 3.1 – 3.2)

Problems for study.

Lipschutz : 3.2 – 3.4, 3.35 – 3.36, 3.39 – 3.40

Exercises to work.

1. Prove the following identities for Cartesian products:

(1) $A \times (B \cap D) = (A \times B) \cap (A \times D)$

(2) $A \times (B \cup D) = (A \times B) \cup (A \times D)$

(3) $A \times (Y - D) = (A \times Y) - (A \times D)$

(4) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(5) $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$

(6) $(X \times Y) - (A \times B) = [X \times (Y - B)] \cup [(X - A) \times Y]$

2. (Part of Halmos, p. 25) Let **A** and **B** be sets. Explain why $A \times B$ is empty if either **A** is empty or **B** is empty.

3. Suppose that **A** and **B** are sets such that $(A \times B) \cap (B \times A)$ is empty. What conclusion can be drawn regarding **A** and **B?**

III.3 : Larger constructions

(Halmos, §§ 3, 5 – 6, 9**;** Lipschutz, §§ 1.9, 3.1 – 3.2, 5.1 – 5.2**)**

Problems for study.

Lipschutz : 1.66(*b*), 3.35, 5.1, 5.3, 5.28, 5.31

Exercises to work.

1. Let **F** be the family of all closed intervals on the real line having the form **[–M, M],** where a real number **x** belongs to **[–M, M]** if and only if the absolute value of **x** is less than or equal to the positive real number **M.** Describe $\$(F)$ and $\cap \{B \mid B \in F\}$. What happens if instead we look at the family of ***open*** intervals having the form **(–M, M),** where a real number **x** belongs to **(–M, M)** if and only if the absolute value of **x** is strictly less than the positive real number **M?**

2. For each integer $n > 1$ let $L(n)$ be the set of all real numbers x such that $x^n < x$, and let F be the family of all subsets having the form $L(n)$ for some $n > 1$. Describe $\$(F)$ and $\cap \{B \mid B \in F\}$.

3. (Halmos, p. 20) Prove that power set construction satisfies the algebraic conditions $P(A) \cap P(B) = P(A \cap B)$ and $P(A) \cup P(B) \subset P(A \cup B)$. More generally, prove that

$$\cap_{A \in C} P(A) = P(\cap_{A \in C} A) \text{ if } C \text{ is nonempty and}$$

$$\cup_{A \in C} P(A) \subset P(\cup_{A \in C} A) \text{ if } C \text{ is arbitrary.}$$

In the case of unions, give an example for which the containment is proper.

4. (Rosen, Exercise 16, p. 85) Can we conclude that $A = B$ if $P(A) = P(B)$? Give reasons for your answer.

> ***Defining ordered triples.*** A method for defining ordered pairs of sets formally is described on page 23 of Halmos (see the bottom half of the page), and the next few pages contain remarks about the definition. Specifically, Halmos defines the ordered pair (a, b) to be $\{\{a\}, \{a, b\}\}$. One can use this to define an ***ordered triple*** (a, b, c) in terms of ordered pairs by the rule
>
> $$(a, b, c) = ((a, b), c).$$

5. Explain why $(a, b, c) = (x, y, z)$ if and only if $a = x$, $b = y$ and $c = z$.

6. Prove that the construction $\langle a, b, c \rangle = \{\{a\}, \{a, b\}, \{a, b, c\}\}$ does ***not*** satisfy the property of ordered triples established in the preceding exercise. **[*Hint*:** One can show that the property holds if $a = b = c$ or all three of a, b, c are distinct, so the search for counterexamples should begin by looking at cases where exactly two of the sets are equal.**]**

III.4 : A convenient assumption

(Halmos, §§ 2; Lipschutz, § 1.12)

Exercises to work.

1. The Axiom of Foundation implies that one cannot have an infinite sequence of sets x_k (for $k \geq 0$) such that $x_k \in x_{k-1}$ for all $k > 0$. We shall say that a nonempty set x has ***Russell type*** equal to n if there exists a sequence of the form

$$x_n \in x_{n-1} \in \ldots \in x_1 \in x$$

but there are no sequences such that $y_{n+1} \in \ldots \in y_1 \in x$. By definition, the empty set will have Russell type 0. Prove that for each nonnegative integer **n** there is a set with Russell type **n.** **[*Hint*:** If a set **x** has Russell type equal to **k,** what is the Russell type of **{x}?]**

> ***Note.*** One can extend this to say that a set has *infinite Russell type* if for each **n** there is a sequence $x_n \in x_{n-1} \in \ldots \in x_1 \in x$. On the basis of what we have done thus far, we cannot conclude that such sets exist, but this will follow from the material in Unit **V.**

2. Let us say that a set has *finite Russell type* if it has Russell type **n** for some nonnegative integer **n.** Prove that the union of two sets with finite Russell type also has finite Russell type.

3. Suppose that we ***define*** the ordered pair **(x, y)** to be the set **{ {x}, {x, y} }** as in pages 23 – 25 of Halmos (and elsewhere**!**) rather than assume its existence. Prove that if **A** and **B** are two sets with finite Russell type, then their Cartesian product also has finite Russell type.

4. If the set **A** has finite Russell type, is the same true of the power set **P(A)?** If this is true and the Russell type of **A** is **n,** then what is the Russell type of **P(A)?**