Exercises for Unit IV (Relations and functions)

IV.1 : Binary relations

(Halmos, § 6**;** Lipschutz, §§ 3.3 – 3.9, 3.11, 7.1 – 7.6, 7.8**)**

Problems for study.

Lipschutz : 3.6(*a*), 3.7(*b*), 3.11, 3.12(*b*), 3.13 – 3.14, 3.16 – 3.18, 3.23 – 3.25, 3.29 – 3.30, 3.32 – 3.33, 3.41(*ab*), 3.45 – 3.50, 3.55, 3.57.

Exercises to work.

1. (Rosen, Exercise 3, p. 480) Determine whether each of the following relations on the set $\{1, 2, 3, 4\}$ is reflexive, symmetric, antisymmetric or transitive.

- **(a)** $\{ (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4) \}$
- **(b)** $\{ (1, 1), (2, 2), (2, 1), (1, 2), (3, 3), (4, 4) \}$
- **(c)** $\{ (2, 4), (4, 2) \}$
- **(d)** $\{ (1, 2), (2, 3), (3, 4) \}$
- **(e)** $\{ (1, 1), (2, 2), (3, 3), (3, 4) \}$
- **(f)** $\{ (1, 3), (1, 4), (2, 3), (2, 4), (3, 1), (3, 4) \}$

2. (Taken from Rosen, Exercise 6, p. 480) Determine whether the relations described by the conditions below are reflexive, symmetric, antisymmetric or transitive.

- **(c)** All ordered pairs of real numbers (x, y) such that $x - y$ is rational.
- **(d)** All ordered pairs of real numbers (x, y) such that $x = 2y$.
- **(e)** All ordered pairs of real numbers (x, y) such that $xy \geq 0$.
- **(f)** All ordered pairs of real numbers (x, y) such that $xy = 0$.

3. (Taken from Rosen, Exercise 7, p. 480) Determine whether the relations described by the conditions below are reflexive, symmetric, antisymmetric or transitive.

- **(a)** All ordered pairs of real numbers (x, y) such that $x \neq y$.
- **(b)** All ordered pairs of real numbers (x, y) such that $xy \geq 1$.
- **(c)** All ordered pairs of real numbers (x, y) such that $x = y \pm 1$.
- **(g)** All ordered pairs of real numbers (x, y) such that $x = y^2$.
- **(h)** All ordered pairs of real numbers (x, y) such that $x \geq y^2$.

4. (Taken from Rosen, Exercise 7, p. 533) Suppose that R_1 and R_2 are reflexive relations on a set **A.** Are their union and intersection reflexive? Give reasons for your answer.

5. (Taken from Rosen, Exercise 1, p. 513) Which of the relations described below on the set $\{0, 1, 2, 3\}$ are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a) $\{ (0, 0), (1, 1), (2, 2), (3, 3) \}$
(b) $\{ (0, 0), (0, 2), (2, 0), (2, 2), (2, 3), (3, 2), (3, 3) \}$
(c) $\{ (0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3) \}$
(d) $\{ (0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3) \}$
(e) $\{ (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 2), (3, 3) \}$

6. (Taken from Rosen, Exercise 2, p. 513) Which of the relations described below on the set of all people are equivalence relations? Determine the properties of an equivalence relation that the others lack.

(a) All **a** and **b** such that **a** and **b** have the same age.
(b) All **a** and **b** such that **a** and **b** have the same parents.
(c) All **a** and **b** such that **a** and **b** have a common parent.
(d) All **a** and **b** such that **a** and **b** have met.
(e) All **a** and **b** such that **a** and **b** speak a common language.

7. Let ***R*** be a binary relation that is reflexive and transitive, and define a new binary relation ***S*** such that **x *S* y** if and only if **x *R* y** and **y *R* x.** Prove that ***S*** is an equivalence relation.

8. (Taken from Rosen, Exercise 10, p. 533) A relation ***R*** is said to be ***circular*** if it satisfies **a *R* b** and **b *R* c** imply **c *R* a.** Show that ***R*** is an equivalence relation if and only if it is reflexive and circular.

9. Let $\mathbf{R}$ denote the real numbers, and let ***P*** be the binary relation on $\mathbf{R} \times (\mathbf{R} - \{0\})$ such that **(x, y) *P* (z, w)** if and only if $xw = yz$. Prove that ***P*** is an equivalence relation, and show that every equivalence class contains a unique element (or representative) of the form $(r, 1)$.

10. Let $\mathbf{N}^+$ be the set of all positive integers, and define a binary relation ***Q*** on the set $\mathbf{N}^+ \times \mathbf{N}^+$ such that **(x, y) *Q* (z, w)** if and only if $x^w = z^y$. Determine whether ***Q*** is an equivalence relation.

11. Let ***R*** be the binary relation in Algebraic Example 3 (the set is a chessboard, and the relation is that two squares are related if there is a knight's move from one to the other).

(i) Let ***E*** be the equivalence relation generated by ***R.*** Show that ***E*** contains exactly one equivalence class; in other words, starting from the standard position of $(1, 2)$ the knight can reach every point on the chessboard.

(ii) Suppose that we replace our 8×8 chessboard with an infinite board whose elements are ordered pairs of integers. Prove that in this case the

equivalence relation E generated by R also has one point. **[*Hint*:** Start at the origin, and show that every adjacent square is E – related to it.**]**

12. (Taken from Rosen, Exercise 38, p. 481) Let R_1 be the relation "**a** divides **b**" on the positive integers, and let R_2 be the relation "**a** is a multiple of **b**" on positive integers. Describe the relations $R_1 \cup R_2$ and $R_1 \cap R_2$.

13. Given two binary relations S and T on a set **A,** their composite $S \circ T$ is defined to be all $(x, z) \in A \times A$ such that there is some $y \in A$ for which $x \, S \, y$ and $y \, T \, z$. **If A** is the real line and S and T are the relations

$$u \, S \, v \text{ if and only if } |u| = |v|$$
$$u \, T \, v \text{ if and only if } |u + 1| = |v - 2|$$

then find all real numbers **z** such that $1 \, S \circ T \, z$ and all real number **z** such that $2 \, S \circ T \, z$. **[*Hint:*** the first relation amounts to saying that $v = au$ where $a = \pm 1$, and the second amounts to saying that $v - 2 = b(u - 1)$ where once again $b = \pm 1$. Why does this imply that there are at most ***four*** choices for **z** in each case**?]**

14. Let S, T_1 and T_2 be binary relations on a set **A.** Prove that we have $S \circ (T_1 \cup T_2) = (S \circ T_1) \cup (S \circ T_2)$ and $S \circ (T_1 \cap T_2) \subset (S \circ T_1) \cap (S \circ T_2)$. Find an example where the inclusion in the second statement is proper. **[*Hint:*** There is an example for which **A** has four elements**.]**

IV.2 : Partial and linear orderings

(Halmos, § 14**;** Lipschutz, §§ 3.10, 7.1 – 7.6**)**

Problems for study.

Lipschutz : 7.1, 7.3 – 7.5, 7.9, 7.11, 7.27(*b*), 7.41 – 7.42, 7.52.

Exercises to work.

1. Let P_1 and P_2 be partial orderings on the set **A.** Prove that $P_1 \cap P_2$ is a partial ordering, and give an example to show that $P_1 \cup P_2$ is not necessarily a partial ordering.

2. (Rosen, Exercise 10, p. 528) Let $S = \{1, 2, 3, 4\}$ with the usual ordering, and take the lexicographic ordering on $S \times S$. Find all elements of $S \times S$ which are less than $(2, 3)$, and find all elements of $S \times S$ which are greater than $(3, 1)$.

3. Let **J** be the set of closed intervals in the real numbers, and define a binary relation P such that **[a,b]** P **[c,d]** if and only if **[a,b] = [c,d]** or $b < c$.

(1) Show that ***P*** defines a partial ordering on **J.**

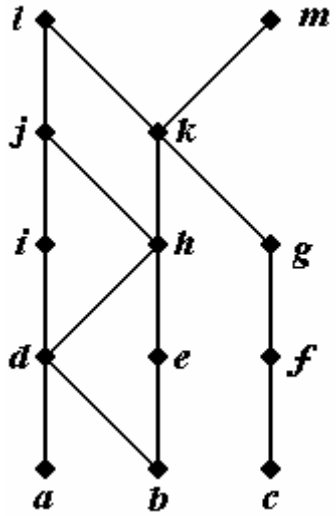
(2) Show that two elements of ***P*** are comparable if and only if they are equal or disjoint.

(3) Show that ***P*** is not a linear ordering on **J.**

4. Let **S** be the set $\{1, \ldots, n\}$. Prove that **P(S)** has a linearly ordered subset **T** with $n + 1$ elements but **S** does not contain a linearly ordered subset with $n + 2$ elements.

5. Let **R[t]** be the set of all polynomials with real coefficients, and define $p \leq q$ if and only if $p(x) \leq q(x)$ for all real values of **x.** Prove this defines a partial ordering of **R[t],** but that this partial ordering is not a linear ordering.

6. (Taken from Rosen, Exercise 26, pp. 528 – 529) Answer these questions for the partially ordered set represented by following *Hasse diagram* (the latter is defined with an example on page 170 of Lipschutz):



(a) Find the maximal elements.
(b) Find the minimal elements.
(c) Is there a greatest element?
(d) Is there a least element?
(e) Find all upper bounds of **{a, b, c}**.
(f) Find the least upper bound of **{a, b, c}** if it exists.
(g) Find all lower bounds of **{f, g, h}**.
(h) Find the greatest lower bound of **{f, g, h}** if it exists.

7. Let $(X, \leq)$ be a linearly ordered set, and let **a, b, c** be three ***distinct*** elements of **X.** We shall say that **b** *is between* **a** *and* **c** if either $a < b < c$ or $c < b < a$ is true**.** Explain why if **b** is between **a** and **c,** then **b** is also between **c** and **a,** and also prove that if $(X, \leq)$ is a linearly ordered set such that **x, y, z** are three distinct elements of **X,** then one and only one of these elements is between the other two.

IV.3 : Functions

(Halmos, §§ 9 – 10**;** Lipschutz, §§ 3.3 – 3.9, 4.1 – 4.4, 5.6, 5.8**)**

Problems for study.

Lipschutz : 4.1 – 4.2, 4.3(*ac*), 4.7, 4.8, 4.33, 4.35, 4.37.

Exercises to work.

1. Let **P** be the set of all U. S. presidents, and let **G** be the set of all ordered pairs **(a, b)** in $\mathbf{P} \times \mathbf{P}$ such that **b** succeeded **a** in office. Is **G** the graph of a function? Explain your answer.

2. Let **A** and **x** be sets. Prove that there is a unique function from **A** to $\{\mathbf{x}\}$. It might be helpful to split the proof into two cases depending upon whether or not **A** is empty.

3. (Halmos, p. 33) Prove that for each set **X** there is a unique function from the empty set to **X,** regardless of whether or not **X** is nonempty. Also prove that there are no functions from **X** to the empty set if **X** is nonempty.

4. (Taken from Rosen, Exercise 4, p. 108) Find the domain and range of the function which assigns to each nonnegative integer its last digit.

5. (Taken from Rosen, Exercise 7, p. 109) Find the domain and range of the function which assigns to each positive integer the number of digits $\mathbf{1, 2, 3, 4, 5, 6, 7, 8, 9}$ that do not appear in the base $\mathbf{10}$ decimal expansion of the integer.

6. Let $\mathbf{f : R \to R}$ be the function $\mathbf{f(x) = 3x - 7.}$ Compute the following sets:

(i) $\mathbf{f^{-1}[\{3\}]}$
(ii) $\mathbf{f[\{5\}]}$
(iii) $\mathbf{f^{-1}[[-7, 2]]}$
(iv) $\mathbf{f[[2, 6]]}$
(v) $\mathbf{f[\varnothing]}$
(vi) $\mathbf{f^{-1}[[3, 5] \cup [8, 10]]}$

7. Let $\mathbf{f : R \to R}$ be the function $\mathbf{f(x) = (x + 1)^2.}$ Compute the following sets:

(i) $\mathbf{f[\{-1\}]}$
(ii) $\mathbf{f^{-1}[[0,1]]}$
(iii) $\mathbf{f^{-1}[[-1, 1]]}$
(iv) $\mathbf{f[[-1, -1] \cup [1, 3]]}$
(v) $\mathbf{f[f^{-1}[[-3, -1]]]}$
(vi) $\mathbf{f[f^{-1}[[-1, 1]]]}$

IV.4 : Composite and inverse functions

(Halmos, § 10**;** Lipschutz, §§ 4.3 – 4.4, 5.7**)**

Problems for study.

Lipschutz : 4.11 – 4.14, 4.17 – 4.18, 4.20, 4.22, 4.39(*b*), 4.45 – 4.46.

Exercises to work.

1. Suppose that **X** is linearly ordered set, **Y** is a partially ordered set, and $\mathbf{f : X \to Y}$ is strictly increasing (***i.e.*****, a < b** implies **f(a) < f(b)** for all **a, b**). Prove that **f** is $\mathbf{1 - 1.}$ Give an example to show that this fails if **X** is not linearly ordered.

2. Suppose that **A** and **B** are sets. Show that the mapping

$$\mathbf{h : P(A) \times P(B) \to P(A \times B)}$$

which sends **(C, D)** to $\mathbf{C \times D \subset A \times B}$ is $\mathbf{1 - 1,}$ and give an example to show it is not necessarily onto.

3. (Taken from Rosen, Exercise 10, p. 109) Determine whether each of the following functions from the set **{a, b, c, d}** to itself is injective.

(a) The function sending the ordered quadruple **(a, b, c, d)** to **(b, a, c, d).**
(b) The function sending the ordered quadruple **(a, b, c, d)** to **(b, b, d, c)**.
(c) The function sending the ordered quadruple **(a, b, c, d)** to **(d, b, c, d)**.

4. (Taken from Rosen, Exercise 18, p. 109) Determine which of these functions are bijections from the set of real numbers to itself**.**

(a) $\mathbf{f(x) = -3x + 4.}$
(b) $\mathbf{f(x) = -3x^2 + 7.}$
(c) $\mathbf{f(x) = (x + 1)/(x + 2)}$ **.**
(d) $\mathbf{f(x) = x^5 + 1.}$

5. A function $\mathbf{f : A \to B}$ is called a ***formal monomorphism*** if for all functions $\mathbf{g, h : C \to A,}$ the equation $\mathbf{fg = fh}$ implies $\mathbf{g = h.}$ Prove that **f** is a formal monomorphism if and only if **f** is injective.

6. Similarly, a function $\mathbf{f : A \to B}$ is called a ***formal epimorphism*** if for all functions $\mathbf{g, h : B \to D,}$ the equation $\mathbf{gf = hf}$ implies $\mathbf{g = h.}$ Prove that **f** is a formal epimorphism if and only if **f** is surjective**.**

7. (Halmos, p. 41) Let **X** and **Y** be nonempty sets, and let $f : X \to Y$ be a function.

- **(a)** Prove that $f(A \cap B) = f(A) \cap f(B)$ for all subsets **A** and **B** of **X** if and only if **f** is $1 - 1$.
- **(b)** Prove that $f(X - A) \subset Y - f(A)$ for all subsets **A** of **X** if and only if **f** is $1 - 1$.
- **(c)** Prove that $Y - f(A) \subset f(X - A)$ for all subsets **A** of **X** if and only if **f** is onto.

8. Let **A** and **B** be nonempty sets, and let $f : A \to B$ be a $1 - 1$ function. Prove that there is a one – sided inverse $g : B \to A$; *i.e.*, we have $gf = id_A$. [***Hint*:** Given an element $z \in A$, define **g** as follows: If $b \in B$ can be written as **f(a)** for some **a**, then set $g(b) = a$; this is well – defined because **f** is injective. Otherwise, let $g(b) = z$.]

9. A function $f : A \to B$ is called a ***retract*** if there is a function $g : B \to A$ such that $gf = 1_A$. Prove that every retract is a monomorphism (this is a converse to a previous exercise). Also prove that the associated map **g** is an epimorphism.

10. A function $f : A \to B$ is called a ***retraction*** if there is a function $g : B \to A$ such that $fg = 1_B$. Prove that every retraction is an epimorphism (this is a converse to a previous exercise). Also prove that **g** is a monomorphism.

11. Let $[0, 1]$ be the closed unit interval, and let **a** and **b** be real numbers which satisfy $a < b$. Construct a bijection from $[0, 1]$ to **[a, b].** Is it unique?

12. Give examples of composable functions **f** and **g** such that **g f** is a bijection but neither **f** nor **g** is a bijection. If **g f** is a bijection, is either of **g** or **f** an injection or a surjection? The preceding question has four separate parts.

13. Find the inverse functions to $p(x) = 3x - 1$ and $q(x) = x/(1 + |x|)$, where the domains of both functions are the real numbers, the codomain of **p** is also the reals, and the codomain of **q** is $(-1, 1)$. [***Hint*:** In the second example it is useful to consider two cases depending upon whether $x \geq 0$ or $x \leq 0$.]

14. (Rosen, Example 24, pp. 107 – 108) For each real number **x,** let $\mathbf{int}(x)$ be the greatest integer that is less than or equal to **x.** Prove that

$$\mathbf{int}(2x) = \mathbf{int}(x) + \mathbf{int}(x + ½).$$

15. (Rosen, Exercise 67, p. 111) Prove or disprove the following statements:

- **(1)** For all **x** and **y,** $\mathbf{int}(x + y) = \mathbf{int}(x) + \mathbf{int}(y)$
- **(2)** For all **x** and **y,** $\mathbf{int}(x) + \mathbf{int}(y) + \mathbf{int}(x + y) = \mathbf{int}(2x) + \mathbf{int}(2y)$

16. (Rosen, Exercise 68, p. 111) Prove that

$$\mathbf{int}(3x) = \mathbf{int}(x) + \mathbf{int}(x + ⅓) + \mathbf{int}(x + ⅔).$$

17. (Taken from Rosen, Exercise 20, p. 109) Let **f** be a function on the real numbers which is positive valued and let $\mathbf{g} = 1/\mathbf{f}$. Explain why **f** is strictly increasing if and only if **g** is strictly decreasing.

18. Prove the **Complement to Proposition IV.4.5** that was stated without proof in the notes: *Suppose we have a function* $\mathbf{f : A \to B}$ *and two factorizations of* **f** *as* $\mathbf{j_0 \circ q_0}$ *and* $\mathbf{j_1 \circ q_1}$ *where the maps* $\mathbf{q_t}$ *are surjective and the maps* $\mathbf{j_t}$ *are injective for* $\mathbf{t = 0, 1}$. *Denote the codomain of* $\mathbf{q_t}$ *(equivalently, the domain of* $\mathbf{j_t}$*) by* $\mathbf{C_t}$. *Then there is a unique bijection* $\mathbf{H: C_0 \to C_1}$ *such that* $\mathbf{H q_0 = q_1}$ *and* $\mathbf{j_1 H = j_0}$. — [***Hint*:** The mapping **H** should be defined so that if $\mathbf{y = q_0(x)}$, then $\mathbf{H(y) = q_1(x)}$. One major step is to show this is well – defined; *i.e.*, if $\mathbf{q_0(x) = q_0(w)}$, then $\mathbf{q_1(x) = q_1(w)}$. This is one place in the proof where the factorization assumptions play an important role. The next steps are to show that **H** is injective and surjective. Finally, it is necessary to prove the uniqueness of **H.**]

IV.5 : Constructions involving functions

(Halmos, § 8; Lipschutz, § 5.7)

Problems for study.

Lipschutz : 5.19, 5.49.

Exercises to work.

1. Given two equivalence relations $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ on a set **X,** let $\mathbf{G_1}$ and $\mathbf{G_2}$ be the partitions of **X** that they determine. The ***cross partition*** $\mathbf{G_{1,2}}$ is the partition whose equivalence classes have the form $\mathbf{C \cap D}$, where **C** is an equivalence class of $\mathbf{G_1}$ and **D** is an equivalence class of $\mathbf{G_2}$. Let $\mathbf{p_1 : X \to X/R_1}$ and $\mathbf{p_2 : X \to X/R_2}$ be the equivalence class projections, and let $\mathbf{q : X \to (X/R_1) \times (X/R_2)}$ be the map such that the coordinates of $\mathbf{q(x)}$ are $\mathbf{p_1(x)}$ and $\mathbf{p_2(x)}$ respectively. Prove that the sets in the cross partition are the inverse images of points under the mapping **q.**

2. Prove the exponential laws stated in **Theorem IV.5.5** of the notes: *If* **A, B** *and* **C** *are sets, then there is a* $\mathbf{1 - 1}$ *correspondence between* $\mathbf{(B \times C)^A}$ *and* $\mathbf{B^A \times C^A}$, *and there is also a* $\mathbf{1 - 1}$ *correspondence between* $\mathbf{(C^B)^A}$ *and* $\mathbf{C^{B \times A}}$. — [***Hints*:** For the first part, let **p** and **q** be the projections from $\mathbf{B \times C}$ to **B** and **C** respectively, then define a map from $\mathbf{(B \times C)^A}$ and $\mathbf{B^A \times C^A}$ sending $\mathbf{f : A \to B \times C}$ to the ordered pair **(p f, q f),** and show this map is a bijection. For the second part, define mappings

$$\mathbf{\Phi : C^{B \times A} \to (C^B)^A}$$

and Ψ in the opposite direction as follows: Given $\mathbf{f : B \times A \to C}$, let $\Gamma \subset \mathbf{(B \times A) \times C}$ be its graph; for each $\mathbf{a \in A}$, explain why there is a unique function $\mathbf{g_a : B \to C}$ whose

graph is equal to $q_{B,C}[\Gamma \cap (B \times \{a\}) \times C]$, where $q_{B,C}$ is the projection from the set $(B \times A) \times C$ onto $B \times C$. Define $\Phi(f) = g$, where the value of **g** at $a \in A$ is the function g_a. Conversely, given $g : A \to C^B$, for each $a \in A$ let $\Gamma(a) \subset (B \times A) \times C$ be the graph of **g(a),** and show that the set Γ, consisting of all

$$((b, a), c) \in (B \times A) \times C$$

such that $(b, c) \in \Gamma(a)$, is the graph of a function $f : B \times A \to C$, and set $\Psi(g)$ equal to **f.** Complete the proof by checking that $\Psi\Phi(f) = f$ and $\Phi\Psi(g) = g$ for all **f** and **g.]**

IV.6 : Order types

(Halmos, § 18; Lipschutz, §§ 7.7 – 7.10)

Problems for study.

Lipschutz : 7.68, 7.73 – 7.74.

Exercises to work.

1. Show that the subset $[0, 1) \cup [2, 3)$ of the real line has the same order type as the half – open interval **[0,2).**

2. Show that the subset $[0, 1] \cup [2, 3]$ of the real line does not have the same order type as the half – open interval **[0, 2].**

3. Let **X** be the power set of an infinite set, and let **Y** be the set of real polynomials with the partial ordering discussed previously, so that both **X** and **Y** are infinite partially ordered sets that are not linearly ordered. Prove that **Y** has the self – density property but **X** does not.

4. Let **D(n)** denote the partially ordered set of positive integers **d** which divide **n,** and take the divisibility relation $a \mid b$ to be the partial ordering. Prove that **D(28)** and **D(45)** are order – isomorphic, but the sets **D(8)** and **D(15)** are not even though they have the same numbers of elements.

5. Let $\mathbf{N}$ be the nonnegative integers with the usual ordering, and take the lexicographic ordering on $\mathbf{N} \times \mathbf{N}$. Prove that the linearly ordered sets $\mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$ have different order types. **[*Hint:*** For each $x \in \mathbf{N}$, the set of all **y** such that $y < x$ is finite**.]**