Exercises for Unit V (The basic number systems of mathematics)

V.1 : The natural numbers and the integers

(Halmos, §§ 11 – 13**;** Lipschutz, §§ 2.1, 2.7 – 2.9**)**

Problems for study.

Lipschutz : 2.88, 2.92(*c*)

Exercises to work.

1. Suppose we are given a quadratic equation $\mathbf{x^2 + b\,x + c = 0}$ where **b** and **c** are integers, and suppose that **r** is a rational root of this equation. Prove that **r** is an integer. **[*Hint*:** Write the quadratic polynomial as **(x – r)(x – s)** and explain why **r + s** and **rs** must be integers. Why does this imply that **s** is also rational? Next, using the quadratic formula show that $\mathbf{(r - s)^2 = b^2 - 4c}$ and hence the right hand side is a is a perfect square, say $\mathbf{d^2}$**.** Next, write **r = p**/**q** where **p** and **q** are relatively prime, and apply the quadratic formula to show that the absolute value of the denominator |**q**| is at most **2.** Why do this and the integrality of **rs** imply that **b** and **d** must be both even or both odd? Now use the quadratic formula once more to show that the root **r** must be an integer**;** ***i.e.*****,** we must have **|q| = 1.]**

2. Explain how one can prove irrationality of the square root of **2** from a special case of the preceding result.

3. Let **n** be a positive integer. Explain why the set **A** of all integers greater than or equal to **– n** is well – ordered. **[*Hint*:** If **B** is a nonempty subset of **A,** consider the set **C** of all integers of the form **n + b** where **b ∈ B.]**

IV.2 : Finite induction and recursion

(Halmos, §§ 11 – 13**;** Lipschutz, §§ 1.11, 4.6, 11.1 – 11.7**)**

Problems for study.

Lipschutz : 1.36, 1.74 – 1.76

Exercises to work.

1. Prove by induction that for each nonnegative integer **k,** the integer $k^2 + 5k$ is even**.**

2. Prove the summation formula $1^3 + \ldots n^3 = (1 + \ldots + n)^2 = n^2 (n + 1)^2 / 4$**.**

3. The number **n!** (**n** *factorial*), which is the number of permutations of the first **n** positive integers, is defined recursively by $0! = 1$ and $n! = n \cdot (n - 1)!$ for all $n > 0$**.** Prove that $n! \leq n^n$ for all $n > 0$ and strict inequality holds if $n > 1$**.**

4. The so – called sequence of ***Fibonacci numbers*** is given recursively by the formulas $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n > 1$**.** Find a function **H** as in the recursive definition theorem which can be used to define the sequence of Fibonacci numbers. **[*Hint*:** The cases $n = 1$ and $n > 1$ must be handled separately.**]**

> ***Comment:*** Fibonacci **(*c*.** 1170 – 1250**),** whose real name was **Leonardo of Pisa,** made many extremely significant contributions to mathematics, but it is ironic that he is often best known today for mentioning a sequence that had been previously introduced by others and which really plays an extremely minor part in his work as a whole. Further information on these historical issues is given at the following online site:
>
> http://math.ucr.edu/~res/math153/history07.pdf

5. An ***amortized loan*** is paid off in equal periodic payments of **P** units. Assume that the periodic interest rate over the time period is $(100 \cdot r)\%$**.** If **L** is the loan amount and x_n is the balance remaining after the n^{th} payment, give a recursive formula for x_n**.**

> ***Comment:*** Usually one is given **L** and **r,** and the objective is to find a value of **P** such that the balance is equal to zero after the M^{th} payment for some fixed **M** (which is normally a multiple of 12)**.** In order to do this, one needs to derive a closed formula for x_n and solve for **P** in terms of **L, r** and **M.**

6. Let (N, σ) be a system satisfying the Peano axioms with zero element 0_N**.** Prove that 0_N is the only element that is not in the image of σ**. [*Hint*:** Apply the third Peano axiom fo the set $A = \{0_N\} \cup \sigma(N)$**.]**

IV.3 : Finite sets

(Halmos, §§ 11 – 13**;** Lipschutz, §§ 1.8, 3.2**)**

Problems for study.

Lipschutz : 1.25, 1.26

Exercises to work.

1. Suppose that we are given two finite sets **A** and **B** and a subset **C** of $\mathbf{A} \times \mathbf{B}$ such that the following hold:

(1) Each element of **A** is the first coordinate for some element of **C.**

(2) For each $\mathbf{a} \in \mathbf{A}$, the number of elements in $\mathbf{C} \cap [\{\mathbf{a}\} \times \mathbf{B}]$ is equal to a fixed constant **k.**

Prove that the number of elements in **C** is equal to $\mathbf{k} \cdot |\mathbf{A}|$, where $|\mathbf{A}|$ denotes the number of elements in **A. [*Hint*:** Use induction on the number of elements in **A.]**

2. Use the preceding exercise to compute the number of ordered pairs **(x, y)** where **x** and **y** are integers between **1** and **10** such that one is even and the other is odd.

3. Determine the number of Boolean subalgebras of **P(X)** if **X** is the set $\{1, 2, 3, 4\}$. How many have exactly **two** atomic subsets?

IV.4 : The real numbers

(Lipschutz, §§ 2.2 – 2.6, 7.7**)**

Problems for study.

Lipschutz : 2.14 – 2.15, 2.17, 2.25 – 2.28, 2.61 – 2.62, 2.67, 2.71 – 2.72, 2.73(*d*), 7.22, 7.25 – 7.26, 7.62 – 7.63, 7.66

Exercises to work.

1. Let **A** be a set of real numbers containing exactly two elements. Explain why **A** is bounded, find the least upper bound and greatest lower bound, and prove that your answers are correct.

2. Let **A** and **B** be a subsets of the real numbers with least upper bounds **u** and **v.** Prove that their union has a least upper bound, and express it in terms of **u** and **v.**

3. Let **A** be the set of negative real numbers. Prove that **0** is equal to the least upper bound of **A. [*Hint*:** One needs to check that **0** is an upper bound and if $x < 0$ then **0** is not an upper bound**;** ***i.e.*****,** there is some $y \in A$ such that $x < y$.**]**

4. Let **A** be a set of real numbers with least upper bound **x.** Show that there is a sequence of elements x_n in **A** whose limit is equal to **x;** note that the terms of a sequence need not be distinct**. [*Hint*:** Given a positive integer **n,** why is there an element **y** of **A** such that $y > a - (1/n)$**?]**

IV.5 : Familiar properties of the real numbers

(Lipschutz, §§ 2.2, 4.5**)**

Problems for study.

Lipschutz : 4.55, 6.17

Exercises to work.

1. Suppose that **a** and **b** are real numbers such that $a < b$. Prove that there are infinitely many rational numbers in the open interval **(a, b).**

2. For an arbitrary base **N,** one has "base **N** decimal – like" expansions analogous to those for base **10.** In particular, if $k > 1$ is a positive integer then such an expansion

$$1/k = (0.x_1x_2x_3\ldots)_N$$

Is given recursively by long division formulas as in the case $N = 10$**:**

$$N = x_1 k + y_1$$

$$Ny_1 = x_2 k + y_2$$

$$\ldots$$

Suppose that $N = 16$ and we write the basic hexadecimal digits in the standard form:

1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F

Find the base **16** decimal – like expansion for $1/k$ where $k = 2, \ldots, 10$.

3. Let **f** be the discontinuous, strictly increasing function from the closed unit interval **[0, 1]** to itself which is defined at the end of the section using decimal expansions, and suppose that **x** is an arbitrary point in the unit interval. Explain why **x** is rational if and only if **f(x)** is rational.

4. In the middle to late 14th century Robert Swineshead (or Suiseth, also known as ***the Calculator***) and N. Orseme evaluated the following infinite series (sometimes called the ***Swineshead series***):

$$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots$$

Oresme's technique is equivalent to viewing the latter as a double infinite sum of nonnegative terms

$$\sum a_{i,j}$$

where $i, j \geq 1$. Standard results in the theory of infinite series (first proved rigorously in the 19th century) imply that the sum of such a series does not change if one rearranges the terms or groups the terms in an arbitrary manner. In particular, if one puts the terms into an infinite matrix where the indices represent the row and column numbers as usual, we can find the value of the sum by first adding along the rows and then along the columns, and similarly we can find the value by first adding along the columns and then along the rows:

$$\sum_{i,j\geq 1} a_{i,j} = \sum_{i\geq 1}\sum_{j\geq 1} a_{i,j} = \sum_{j\geq 1}\sum_{i\geq 1} a_{i,j}$$

A proof of these formulas appears in the following online document:

http://www.math.umn.edu/~jodeit/course/PwrThms.pdf

If we let $a_{i,j} = 2^{-(i+j)}$ if $i \leq j$ and 0 otherwise, then if we sum first over j holding i fixed, and then we sum over i, we obtain the series displayed above. On the other hand, if we sum first over i holding j fixed, and then we sum over j, we obtain a precise numerical value for the Swineshead series. What is it? **[*Hint*:** Write out the matrix array $a_{i,j}$ explicitly.**]**