Exercises for Unit VI (Infinite constructions in set theory)

VI.1 : Indexed families and set – theoretic operations

(Halmos, §§ 4, 8 – 9**;** Lipschutz, §§ 5.3 – 5.4**)**

Problems for study.

Lipschutz : 5.3 – 5.6, 5.29 – 5.32, 9.14

Exercises to work.

1. Generalize Exercise **12** from Section **III.1** to unions and intersections of arbitrary indexed families of sets**:** Suppose that we have nonempty indexed families of sets $\{ A_j \mid j \in J \}$ and $\{ C_j \mid j \in J \}$ such that $A_j \subset C_j$ for all **j,** Prove the following relationships:

$$\left(\bigcap_{j \in J} A_j\right) \subset \left(\bigcap_{j \in J} C_j\right)$$

$$\left(\bigcup_{j \in J} A_j\right) \subset \left(\bigcup_{j \in J} C_j\right)$$

2. Generalize DeMorgan's laws to unions and intersections of arbitrary indexed families of sets as follows:. Suppose that **S** is a set and we have a nonempty indexed families of subsets of **S** of the form $\{ A_j \mid j \in J \}$**.** Prove the following identities:

$$S - \bigcup_{j \in J} A_j = \bigcap_{j \in J} (S - A_j)$$

$$S - \bigcap_{j \in J} A_j = \bigcap_{j \in J} (S - A_j)$$

3. (Halmos, p. 35) **(a)** Given that $\{ A_j \mid j \in X \}$ and $\{ B_k \mid k \in Y \}$ are nonempty indexed families of sets, prove the following indexed distributive identities:

$$\left(\bigcup_{j \in J} A_j\right) \cap \left(\bigcup_{k \in K} B_k\right) = \bigcup_{j,k} (A_j \cap B_k)$$

$$\left(\bigcap_{j \in J} A_j\right) \cup \left(\bigcap_{k \in K} B_k\right) = \bigcap_{j,k} (A_j \cup B_k)$$

(b) Suppose that $\{ I_j \mid j \in J \}$ is an indexed family of sets, and write

$$K = \bigcup \{ I_j \mid j \in J \}.$$

Suppose we are also given an indexed family of sets $\{ A_k \mid k \in K \}$**.** Prove the following identities, assuming in the second case that each of the indexed families is nonempty**:**

$$\bigcup_{k \in K} A_k = \bigcup_{j \in J} \left(\bigcup \{ A_i \mid i \in I_j \} \right)$$

$$\cap_{k \in K} A_k = \cap_{j \in J} \left(\cap \{ A_i \mid i \in I_j \} \right)$$

4. (Halmos, p. 37) **(a)** Let $\{ A_j \mid j \in J \}$ and $\{ B_k \mid k \in K \}$ be indexed families of sets. Prove that

$$\left(\cup_{j \in J} A_j \right) \times \left(\cup_{k \in K} B_k \right) = \cup_{j,k} (A_j \times B_k)$$

(another indexed distributive law) and that a similar formula holds for intersections provided that all the indexing sets are nonempty.

(b) Let $\{ X_j \mid j \in J \}$ be an indexed family of sets. Prove that

$$\left(\cap_{j \in J} X_j \right) \subset X_k \subset \left(\cup_{j \in J} X_j \right)$$

for all $k \in J$. Furthermore, if **M** and **N** are sets such that $M \subset X_j \subset N$ for all **j,** prove that

$$M \subset \left(\cap_{j \in J} X_j \right) \quad \text{and} \quad \left(\cup_{j \in J} X_j \right) \subset N.$$

VI.2 : Infinite Cartesian products

(Halmos, § 9**;** Lipschutz, §§ 5.4, 9.2**)**

Problems for study.

Lipschutz : 5.11

Exercises to work.

1. (**"**A product of products is a product.**"**) Let X_j be a family of nonempty sets with indexing set **J,** and let $J = \cup \{ J_k \mid k \in K \}$ be a partition of **J.** Construct a bijective map from $\prod_j X_j$ to the set

$$\prod_{k \in K} \left(\prod \{ X_j \mid j \in J_k \} \right).$$

[*Hint*: Use the Universal Mapping Property.**]**

2. Let **J** be a set, and for each $j \in J$ let $f_j : X_j \to Y_j$ be a set – theoretic map. Prove that there is a unique map

$$F = \prod_j f_j : \prod_j X_j \to \prod_j Y_j$$

defined by the conditions

$$p_j^{Y} \circ F = f_j \circ p_j^{X}$$

where $\mathbf{p_j}^{\mathbf{X}}$ and $\mathbf{p_j}^{\mathbf{Y}}$ denote the $\mathbf{j}^{\mathbf{th}}$ coordinate projections for $\prod_j \mathbf{X_j}$ and $\prod_j \mathbf{Y_j}$ respectively. Also prove that this map is the identity map if each $\mathbf{f_j}$ is an identity map. Finally, if we are also given sets $\mathbf{Z_j}$ with maps $\mathbf{g_j : Y_j \to Z_j}$, and $\mathbf{G = \prod_j g_j}$, then show that $\mathbf{G \circ F = \prod_j (g_j \circ f_j)}$.

> ***Notation.*** The map of products $\prod_j \mathbf{f_j}$ constructed in the preceding exercise is frequently called the ***product*** of the maps $\mathbf{f_j}$.

3. Let $\{\mathbf{X_j}\}$ and $\{\mathbf{Y_j}\}$ be indexed families sets with the same indexing set **J,** and assume that for each $\mathbf{j \in J}$ the mapping $\mathbf{f_j : X_j \to Y_j}$ is a bijection. Prove that the product map $\prod_j \mathbf{f_j} : \prod_j \mathbf{X_j} \to \prod_j \mathbf{Y_j}$ is also a bijection. **[*Hint*:** What happens when one takes the product of the inverse maps?**]**

4. Suppose in the preceding exercise we only know that each mapping $\mathbf{f_j}$ is an injection or each mapping $\mathbf{f_j}$ is a surjection. Is the corresponding statement true for the product map? In each case either prove the answer is yes or find a counterexample.

> ***Coequalizers.*** Here is another fundamental example of a universal mapping property. Given two functions $\mathbf{f, g : A \to B,}$ a ***coequalizer*** of **f** and **g** is defined to be a map $\mathbf{p : B \to C}$ such that $\mathbf{p\,f = p\,g}$ which has the following universality property: Given an arbitrary map $\mathbf{q : B \to D}$ such that $\mathbf{q\,f = q\,g}$, then there exists a unique mapping $\mathbf{h : C \to D}$ such that $\mathbf{q = h\,p.}$ — In geometrical studies, such constructions arise naturally if one tries to build an object out of two simpler pieces by gluing them together in some manner (say along their edges), and there are also numerous other mathematical situations where examples of this concept arise.

5. Prove that every pair of functions $\mathbf{f, g : A \to B}$ has a coequalizer. **[*Hint*:** Consider the equivalence relation generated by requiring that **f(x)** be related to **g(x)** for all **x** in **A.]**

6. In the setting of the previous exercise, suppose that $\mathbf{p : B \to C}$ and $\mathbf{r : B \to E}$ are coequalizers of **f** and **g.** Prove that there is a unique bijection $\mathbf{H : C \to E}$ such that $\mathbf{r = H\,p.}$ **[*Hint*:** Imitate the proof of the corresponding result for products**.]**

VI.3 : Transfinite cardinal numbers

(Halmos, §§ 22 – 23**;** Lipschutz, §§ 6.1 – 6.3, 6.5**)**

Problems for study.

Lipschutz : 6.4, 6.12

Exercises to work.

1. (Halmos, p. 92) Prove that the set **F(S)** of finite subsets of a countable set **S** is countable, and it is (countably) infinite if and only if **S** is (countably) infinite.

2. Suppose that ***E*** is an equivalence relation on a countably infinite set **S,** and let **S/*E*** be the associated family of equivalence classes. Explain why **S/*E*** is countable.

VI.4 : Countable and uncountable sets

(Halmos, §§ 23 – 23; Lipschutz, §§ 6.3 – 6.7)

Problems for study.

Lipschutz : 6.2 – 6.3, 6.14, 6.32

Exercises to work.

1. (Halmos, p. 95) Let $\alpha, \beta, \gamma, \delta$ be cardinal numbers such that $\alpha \leq \beta$ and $\gamma \leq \delta$. Prove that $\alpha + \gamma \leq \beta + \delta$ and $\alpha \cdot \gamma \leq \beta \cdot \delta$.

2. Let $\alpha \neq 0$ be a cardinal number. Prove that $\alpha \cdot 0 = 0$, $\alpha^1 = \alpha$ and $1^\alpha = 1$.

3. Let $\Sigma(\mathbf{R})$ denote the set of all $1-1$ correspondences from the real numbers to itself. Prove that the cardinal number of $\Sigma(\mathbf{R})$ is equal to $2^{|\mathbf{R}|}$. **[*Hint*:** Why is $2^{|\mathbf{R}|}$ equal to $|\mathbf{R}|^{|\mathbf{R}|}$? Why is $\Sigma(\mathbf{R})$ a subset of $\mathbf{R}^{\mathbf{R}}$ and what conclusion does this yield? Next, for each subset of $\mathbf{R}$ define a $1-1$ correspondence from $\mathbf{R}$ to itself as follows: Since we have $|\mathbf{R}| + |\mathbf{R}| = |\mathbf{R}|$, it follows that we can partition $\mathbf{R}$ into two pairwise disjoint subsets **A** and **B** that are each in $1-1$ correspondence with $\mathbf{R}$; let **f** and **g** be $1-1$ correspondences from $\mathbf{R}$ to **A** and **B** respectively. For $\mathbf{C} \subset \mathbf{R}$, define a $1-1$ correspondence $\mathbf{h_C}$ such that $\mathbf{h_C}$ interchanges **f(t)** and **g(t)** for each $\mathbf{t} \in \mathbf{C}$ and $\mathbf{h_C(x)} = \mathbf{x}$ otherwise. Why are $\mathbf{h_C}$ and $\mathbf{h_D}$ unequal if $\mathbf{C} \neq \mathbf{D}$? Look at the set of all $\mathbf{y} \in \mathbf{A}$ such that $\mathbf{h_C(y)} \neq \mathbf{y}$, and use this to conclude that there is a $1-1$ mapping from $\mathbf{P}(\mathbf{R})$ into $\Sigma(\mathbf{R})$.**]**

4. Prove that the set of countable subsets of the real numbers has the same cardinality as the real numbers themselves.

5. It is known that a continuous function on an interval in the real numbers is completely determined by its values at rational points. What does this imply about the cardinal number of continuous functions on an interval?

6. What is the cardinal number of the set of all partial orderings on $\mathbf{N}$ (the nonnegative integers)? **[*Hint:*** There is a $\mathbf{1-1}$ correspondence between binary relations and subsets of $\mathbf{N \times N}$**.** What upper bound does this yield for the set of all partial orderings? For every subset **A** of $\mathbf{N}$ with more than one element, consider the partial ordering which agrees with the usual one on **A** but is modified so that no elements in the complement $\mathbf{N - A}$ are comparable to any other elements in $\mathbf{N}$**.** Why do different subsets determine different partial orderings? Think about the collection of isolated elements that are not comparable to anything other than themselves. How many subsets of this type are there in $\mathbf{N}$**? —** ***Note:*** A considerably more difficult version of this exercise is to show that the cardinality of the set of all partial orderings on $\mathbf{N}$ is equal to the cardinality of the set of all order types of partial orderings on $\mathbf{N}$**.]**

VI.6 : Transfinite induction and recursion

(Halmos, §§ 12 – 13, 17 – 20**;** Lipschutz, §§ 8.1 – 8.9, 8.12 – 8.13**)**

*Problems for study***.**

Lipschutz : 8.21, 8.22

*Exercises to work***.**

1. (Halmos, p. 68) A subset **C** of a partially ordered set **A** is said to be *cofinal* if for each $\mathbf{a \in A}$ there is some $\mathbf{c \in C}$ such that $\mathbf{c \geq a}$**.** Prove that every linearly ordered set has a cofinal well – ordered subset.

2. (Halmos, p. 69) Prove that a linearly ordered set is well – ordered if and only if the set of strict predecessors of each element is well – ordered.

3. Prove that a well – ordered set is finite if and only it is well – ordered with respect to the opposite ordering. **[*Hint*:** An *infinite* well – ordered set must contain a copy of the first infinite ordinal ω**.]**

Exercises for Unit VII (The Axiom of Choice and related topics)

General remark. In all the exercises for this section, the Well – Ordering Principle, the Axiom of Choice, or Zorn's Lemma – or any statement that is shown ***in the course notes*** to follow from these – may be assumed unless explicitly stated otherwise.

VII.1 : Nonconstructive existence statements

(Halmos, §§ 15 – 17**;** Lipschutz, §§ 5.9, 7.6, 9.1 – 9.7**)**

Problems for study.

Lipschutz : 7.16, 9.12

Exercises to work.

1. Use the Axiom of Choice to prove the following statement: If $\mathbf{f : A \to B}$ is a function, then there is a function $\mathbf{g : B \to A}$ such that $\mathbf{f = fgf}$.

2. Let **A** and **B** be sets. Prove that $|\mathbf{A}| \leq |\mathbf{B}|$ if and only if there is a surjection from **B** to **A.** **[** ***Hint*****:** One implication direction is in the notes for this section, and the other is in the exercises for Section **IV.4**. **]**

DEFINITION(S). It is possible to define ***transfinite arithmetic operations*** on cardinal numbers. Specifically, if we are given an indexed family of cardinal numbers α_j (with indexing set **J**) and sets $\mathbf{X_j}$ such that $|\mathbf{X_j}| = \alpha_j$**,** then the *transfinite product* $\prod_j \alpha_j$ is equal to the cardinality of $\prod_j \mathbf{X_j}$**.** According to Exercise **3** in Section **VI.2,** this cardinal number does not depend upon the choice of the indexed family of sets $\mathbf{X_j}$ (assuming these sets satisfy $|\mathbf{X_j}| = \alpha_j$ for all **j**)**.**

We would like to define a corresponding transfinite sum of the cardinal numbers.

Given an indexed family of sets $\mathbf{Y_k}$ with indexing set **K,** the ***disjoint union,*** sometimes also called the ***set – theoretic sum,*** is defined to be the set

$$\coprod_k \mathbf{Y_k} = \{ (y, q) \in (\cup_k \mathbf{Y_k}) \times \mathbf{K} \mid y \in \mathbf{Y_q} \}.$$

3. In the setting above, let **W** be the set defined in the displayed equation, and define $\mathbf{W_q}$ to be the set of all points in **W** whose second coordinate is equal to **q.** Prove

that the sets W_q are disjoint, their union is all of $\coprod_k Y_k$, and for each **q** we have $|W_q| = |Y_q|$.

4. In the setting above, suppose that we are given a second indexed family of sets V_k with the same indexing set **K,** and that for each **k** in **K** we have a bijection f_k from Y_k to V_k. Prove that there is a bijection from $\coprod_k Y_k$ to $\coprod_k V_k$.

> ***Consequence and definition.*** By the conclusion of the preceding exercise, if we are given an indexed family of cardinal numbers α_j as above and we set the *transfinite sum* $\Sigma_j \alpha_j$ equal to the cardinality of the set $\coprod_j X_j$, then this cardinal number does not depend upon the choice of indexed family X_j such that $|X_j| = \alpha_j$.

Footnote. The set – theoretic *sum* or *disjoint union* construction has numerous formal properties that we shall not discuss in this course. Further information may be found in Section **V**.2 of the online notes

http://math.ucr.edu/~res/math205A/gentopnotes.pdf

and the corresponding exercises in the following online document:

http://math.ucr.edu/~res/math205A/gentopexercises.pdf

VII.2 : Extending partial orderings

(Lipschutz, §§ 7.6**)**

Problems for study.

Lipschutz : 7.16 – 7.18

Exercises to work.

1. (Taken from Rosen, Exercise 55, p. 530) Find a compatible linear ordering for the partial ordering in Exercise 26 on pp. 528 – 529 of Rosen (see Exercise **6** for Section **IV.2**).

2. (Rosen, Exercise 56, p. 530) For the partial ordering on the subset of positive integers

{1, 2, 3, 6, 8, 12, 24, 36}

determined by divisibility, find a linear ordering containing it.

3. (Taken from Rosen, Exercise 58, p. 530) Suppose that we are given a set of tasks

A, B, C, D, E, F, G, H, K, L, M

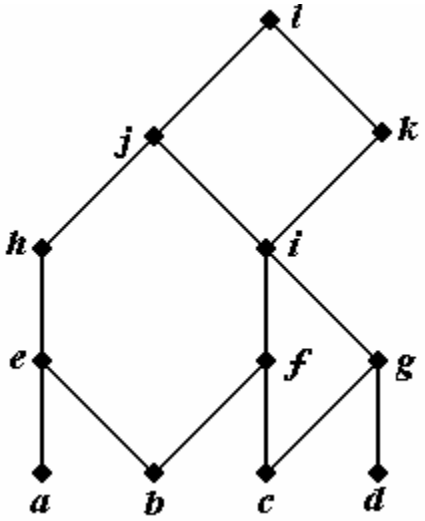
that need to be completed to finish a job, and that they must be scheduled as indicated below:

A	must precede	**B**
B	must precede	**C**
C	must precede	**D**
D	must precede	**E**
E	must precede	**F**
A	must precede	**G**
G	must precede	**H**
G	must precede	**C**
H	must precede	**K**
H	must precede	**D**
K	must precede	**F**
A	must precede	**L**
L	must precede	**M**
M	must precede	**F**

Find a scheduling of the tasks that is compatible with these conditions. **[*Hint :*** View the list as defining a partially ordered set, and draw a Hasse diagram to represent this partially ordered set. Then find a compatible linear ordering for the set**.]**

4. Find a compatible linear ordering for the partially ordered set **P(X),** where **X = {1, 2, 3}.**

5. Find a compatible linear ordering for the partially ordered set with the following Hasse diagram:



6. Suppose that ***L*** is a linear ordering on a set **X** which contains more than two elements. Prove that ***L*** contains a partial ordering ***P*** which is ***not*** a linear ordering. **[*Hint :*** Let **A** be a subset of **X** with three elements; take ***P*** so that no elements in **X – A** are comparable to each other and the restriction **P**|**A** is not a linear ordering.**]**

VII.3 : Equivalence proofs

(Halmos, §§ 15 – 17**;** Lipschutz, §§ 5.9, 7.6, 9.1 – 9.7**)**

Problems for study.

Lipschutz : 7.16, 9.12

Exercises to work.

1. Prove the following result, which is independently due to J. W. Tukey (1915 – 2000) and O. Teichmüller (1913 – 1943), and is generally known as **Tukey's Lemma:** Let **F** be a family of subsets of a fixed set **X,** and assume that it has ***finite character*****;** ***i.e.*****,** a set **A** lies in **F** if and only if every finite subset of **A** lies in **F.** Then **F** has a maximal element. **[*Example*:** The linearly independent subsets of a vector space form a family of finite character.**]**

2. Let **S** be a set, and let $\mathbf{F} \subset \mathbf{P(S)}$ be a collection of pairwise disjoint subsets. Prove that there is a subset **C** of **S** that has ***exactly one element in common with each*** subset **A** in **F.**

VII.4 : Additional consequences

(Halmos, §§ 15 – 17**;** Lipschutz, §§ 5.9, 7.6, 9.1 – 9.7**)**

Problems for study.

Lipschutz : 7.16, 9.12

Exercises to work.

1. (Halmos, p. 95) Let α_j and β_j be indexed families of cardinal numbers with indexing set **J** such that $\alpha_j < \beta_j$ for all $j \in J$. Prove that $\Sigma_j \alpha_j < \Pi_j \beta_j$. **[*Hint*:** For each $j \in J$ let X_j and Y_j be sets such that $|X_j| = \alpha_j$ and $|Y_j| = \beta_j$. It will suffice to show that there is no surjection from $\coprod_j X_j$ to $\Pi_j Y_j$. Use a modified Cantor diagonal process argument to show that any map from the first set to the second is not onto. **]**

2. In the setting of the preceding exercise, what conclusion (if any) can be drawn if the inequalities of cardinal numbers are not necessarily strict and all the cardinal numbers in sight are transfinite? Prove your assertion or give examples**. [*Remark*:** The hypothesis that all cardinal numbers under consideration are infinite is added to

make the proof simpler; it allows one to assume that $|\mathbf{A}| + 1 = |\mathbf{A}|$ for all sets **A** that arise in the discussion**.]**

3. (Halmos, p. 96) Suppose that α, β and γ are cardinal numbers and $\alpha \leq \beta$**.** Prove that $\alpha^\gamma \leq \beta^\gamma$**.** Also prove that if α and β are finite but greater than **1** and γ is infinite, then $\alpha^\gamma = \beta^\gamma$**.**

4. (Halmos, p. 100) If **X** is an infinite set, let λ_0**(X)** be the least ordinal λ such that there is a bijection from λ to **X;** as indicated in the notes, the existence of such an ordinal is a consequence of the Well – Ordering Principle**.** Explain why λ_0**(X)** is a limit ordinal.

5. If we define cardinal numbers to be equal to specific ordinal numbers as in the preceding exercise, which is the first ordinal that is ***not*** equal to a cardinal number?

6. (Halmos, p. 101) If **A** is an infinite set, what is the cardinality of the set of all countable subsets of **A**? **[*Hint*:** There are two cases depending upon whether or not $|\mathbf{A}| > |\mathbf{R}|$**.]**

7. Explain why there is a first ordinal Λ_1 such that $|\Lambda_1| > \aleph_0$**,** and prove that every *countable* set of ordinals in Λ_1 has a least upper bound in Λ_1**.** The latter is often called the ***first uncountable ordinal.***

8. If **S** is a set, then a family of subsets **F** of **S** has the ***finite intersection property*** if for every finite subfamily $\{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$ of **F** the intersection $\cap_j \mathbf{A}_j$ is nonempty. Prove that if **F** has the finite intersection property, then **F** is contained in a maximal family of subsets which has the finite intersection property.