SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 1

Fall 2006

(*Complete through Section* **II.1**)

NOTE. Strictly speaking, the material in the exercises called *Questions to consider* will not be covered on examinations, and similarly for the two groups of exercises for Section II.0 (*Problems from Rosn* and *Additional exercises*), but an understanding of proofs at the level of the exercises for Section II.0 will at least be useful, and it would also be good to understand the responses given here to the questions for Section **II.1**.

I. General considerations

I.1 : Overview of the course

Questions to consider

GENERAL REMARK. There are many possible correct answers to the questions for this section, and the ones given below are just typical examples.

1. No one wants a building to have serious malfunctions when subjected to everyday stresses and strains. Constant repairs of the supporting structure make it impossible to do the work that one needs to do inside the building. Similarly, if mathematical foundations are not constructed carefully, it is far more likely that problems with them will develop all the time and interfere with the attempts to use mathematics for understanding concepts and problems.■

2. Careful preparation and designing often save time and energy in the long run, more than repaying the initial investment needed to set them up and allowing one to do things that would otherwise be very difficult or impossible. The same principle holds for mathematics. Good formulations can make it much easier to understand a problem and solve it efficiently or successfully.■

3. If one dwells to much on small details for their own sake, this can disrupt efforts to understand the original problem.■

4. Crash tests for automobiles are one example as are similar destruction tests to determine the strength of objects like boxes or other containers. Heat tests to determine safe usages for consumer or industrial products are another example.■

I.2 : Historical background and motivation

Questions to consider

1. The integral formula is just an extremely precise and convenient approximation to the physical center of mass.

2. The argument assumes that the angle sum for all triangles is the same number S, and the possibility of different sums for different triangles is overlooked.■

3. The equation of the line has the form $y - \frac{1}{2} = m(x - \frac{1}{2})$. One needs to show that each of these lines, and also the vertical line $x = \frac{1}{2}$ contains either a point with coordinates $(0, t)$ such that $0 \leq t \leq 1$ or a point with coordinates $(t, 0)$ such that $0 \leq t \leq 1$. This can be done using a case by case argument for different choices of m and also for the vertical line. We shall merely list the different cases and describe what happens. Drawing pictures for each case is strongly recommended, and using the pictures one can then proceed to solve equations and prove that the coordinates of the solution have the indicated form. Note that the slope m cannot be equal to -1 because that is the slope of the line joining $(0, 1)$ and $(1, 0)$.

$m = 1$. — This line also goes through the origin, which lies on the triangle.

$m > 1$. — These lines also goes through a point between $(0, 0)$ and $(1, 0)$ of the form $(t, 0)$ where t lies between 0 and $\frac{1}{2}$.

$m = \infty$ (vertical line). — This lines also goes through the point $(\frac{1}{2}, 0)$.

$m < -1$. — These lines also goes through a point of the form $(t, 0)$ where t lies between $\frac{1}{2}$ and 1.

$-1 < m < 1$. — These lines also goes through a point of the form $(0, t)$ where t lies between 0 and 1.■

4. There are four possibilities for the distribution of cards. Let A denote the number of players receiving a picture card from the first deck and a number card from the second, let B be the number of players receiving two picture cares, let C be the number of players receiving a number card from the first deck and a picture card from the second, and let D be the number of players receiving two number cards.

We then have

$$A + B + C + D \quad = \quad 52$$

which is the total number of players and cards in each deck. Since there are 12 picture cards and 40 number cards in each deck we also have

$$A + B \quad = \quad B + C \quad = \quad 12 \qquad\qquad C + D \quad = \quad A + D \quad = \quad 40$$

and since all numbers are nonnegative it follows that $A \leq 12$ and $D \leq 40$. Thus we have

$$D \quad = \quad 40 - A \quad \geq 40 - 12 \quad = \quad 28$$

which is larger than half the number of players.

5. Let x_n denote the n^{th} partial sum of the series. Then

$$s_{3n} \quad = \quad \left(\frac{1}{2} - \frac{1}{4}\right) \; + \; \left(\frac{1}{6} - \frac{1}{8}\right) \; + \; \cdots \; + \; \left(\frac{1}{4n-2} - \frac{1}{4n}\right)$$

so that $\lim_{n\to\infty} s_{3n} = \frac{1}{2}\ln 2$. Also, we have

$$s_{3n+1} \quad = \quad s_{3n} \; + \; \frac{1}{2n+1} \qquad\qquad s_{3n+2} \quad = \quad s_{3n} \; + \; \frac{1}{4n+2}$$

so that $\lim_{n\to\infty} s_{3n+1} = \lim_{n\to\infty} s_{3n} = \lim_{n\to\infty} s_{3n+2}$. We need to piece these together to prove that $\lim_{n\to\infty} s_n = \frac{1}{2}\ln 2$.

We now know the limits of the three sequences with terms s_{3n}, s_{3n+1}, s_{3n+2} exist and have the same value L. Therefore, for each $\varepsilon > 0$ we can find some positive integer M such that $m \geq M$ implies $|y_m - L| < \varepsilon$ for each of the sequences we have considered. Now suppose that $n \geq N$. Write $n = 3m + r$ where m is a nonnegative integer and r is one of 0, 1, 2. If $M \geq 3M + 3$, then $m \geq M$ and hence we have $|s_n - L| = |s_{3m+r} - L| < \varepsilon$. By the previous paragraph we know that $L = \frac{1}{2}\ln 2$, so this completes the proof.■

6. If we substitute $x = \pi/2$ into the series we obtain

$$-\frac{4}{\pi}\cdot(1 \ - \ 3 \ + \ 5 \ \cdots)$$

which is a divergent series and as such does not converge to zero.■

I.3 : Selected problems

Questions to consider

1. It is not possible to impose such conditions because one always can put all the objects into the same box. However, if we limit the number of objects that can be put into a given box, then in some cases we can conclude that some box must contain at least two objects. In the suggested example, suppose that box k contains a_k objects, where we insist that $a_k < 3$ for all k. Then we have $\sum_k a_k = 2n$; if each a_k is less than 2, then since there are n terms it follows that the sum is at most n. Note that this sort of argument actually proves a little more; namely, every box must contain exactly two elements under the extra condition.■

2. This is just an algebraic exercise involving geometric series. The right hand side is equal to

$$\frac{2}{8}\left(1 + \frac{1}{8^2} \ \cdots\right) \ + \ \frac{5}{8^2}\left(1 + \frac{1}{8^2} \ \cdots\right)$$

so all one needs to do is note that the sum of the geometric series which appears twice in this expression is equal to $64/63$ and simplify the resulting expression to check it is equal to $1/3$.■

3. The simplest way to get some insight into y^3 is to cube the equation $x = y - 1$. This yields

$$2 \ = \ x^3 \ = \ y^3 \ - \text{‘}3y^2 \ + \ 3y \ - \ 1$$

which simplifies to $y^3 = 3y^2 - 3y + 3$. If we subtract the expression on the right hand side from both sides of the equation, we obtain a nontrivial cubic polynomial which has y as one of its roots.■

II. Basic concepts

II.0 : Topics from Logic

Problems from Rosen

Some of these are taken fairly directly from two solutions manuals that have been published to be used with that text; the Student Solutions Guide (ISBN 0–07–247477–7) contains solutions for the odd-numbered exercises, and the Instructor's Resource Guide (ISBN 0–07–247480–7) contains solutions for the even-numbered exercises.

1. (*Sketch*) In order to show that one is equivalent to the other, it is necessary to look at all possible cases (p and q are both true, p is true but q is false, p is false but q is true, p and q are both false) and check that the truth values of the two compound statements are the same in all these cases. Direct checking shows that each of the latter will be true in the second and third cases but false in the others.

2. (*Sketch*) Apply the same idea as in the previous exercise to show the equivalence. Verifying that all three operations can be expressed using Sheffer's stroke starts by observing that NOT is so expressible. At the next step we see that AND is expressible in terms of NOT and Sheffer's stroke, so by the first step it can be expressed entirely in terms of the latter. Finally, at the third step we see that OR can be expressed in terms of NOT and AND; since each of the latter can be expressed entirely in terms of Sheffer's stroke, it follows that OR can also be so expressed.

3. An implication is true if the hypothesis is false, so it is easy for the second compound statement to be true if we take $P(x)$ to be any statement that is not always true. For examples, suppose let $P(x)$ denote, "x is an even number." If we now take (x) to be the statement, "x is divisible by 4," then the first compound statement will be false, but the first will be true.

4. Both are true precisely when at least one of the statements $P(x)$ and $Q(x)$ is true for at least one admissible choice of x.

5. It suffices to find a counterexample. Let $P(x)$ be the statement that x is an even number, and let $Q(x)$ be the statement that x is an odd number. Then the first compound statement is true (every number is even or odd) but the second (all numbers are even or all numbers are odd) is false.

6. Take $P(x)$ and $Q(x)$ as in the previous exercise. Then the first statement (there is a number that is both even and odd) is false, but the second (there is an even number and there is an odd number) is true.

7. A less formal way of expressing $P(x, y)$ is to say that student x has taken class y. In these terms, here are the everyday versions of the statements in the exercises:
(a) There is some student who has taken some class.
(b) There is some student who has taken all the classes.
(c) Every student has taken some class.
(d) There is a class that every student has taken.
(e) Every class has been taken by some student.
(f) Every student has taken every class.

8. We shall do them in order.

(a) There is some x such that $x + y = y$ for all y.
(b) For all numbers x and y, if x is nonnegative and y is negative, then $x - y$ is nonnegative.
(c) For all numbers x and y, the product xy is nonzero if and only if both x and y are nonzero.
(d) There are numbers x and y such that $x^2 > y$ and $x < y$.
(e) For all numbers x and y, there is a number z such that $x + y = z$.
(f) For all numbers x and y, if x and y are negative then their product xy is positive.

9. In fact, the first number is not a perfect square, for if it could be written as n^2 for some positive integer n then the rational number $n/10^{250}$ would be the square root of 2. Since $\sqrt{2}$ is irrational, this yields a proof by contradiction. This proof is constructive because we explicitly describe a number from the possibilities in the theorem which is not a perfect square.

10. It suffices to observe that $9 = 3^2$ and $8 = 2^3$ satisfy the given condition.

Note. Recently P. Mihăilescu proved a conjecture made in the 19^{th} century by E. Catalan (1814–1894); namely, that this is the **only** pair of consecutive positive integers which can be expressed as a^x and b^y, where a, b, x, y are all positivie integers and the exponents x, y are greater than 1 (of course, the answer to the problem is no for trivial reaons if we allow either exponent to equal 1). The proof is at a very advanced level, but for the sake of completeness here is a reference: P. Mihăilescu, *Primary cyclotomic units and a proof of Catalan's Conjecture*, [Crelle] Journal für die reine und angewandte Mathematik **572** (2004), 167–195.

11. Each of the three numbers is either nonnegative or nonpositive, so at least two of them, say m and n, are of the same type (positive or negative). But this means their product is nonnegative. This proof is nonconstructive because we are only saying that one of the products is nonnegative and do not specify which one(s) might satisfy the condition. (Actually all three numbers turn out to be positive and hence all the pairwise products are too.)

12. To prove existence, suppose that n is odd, and write it as $2k + 1$ for some other integer k. Then simple algebra shows that n is equal to $(k - 2) + (k + 3)$. — To prove uniqueness, suppose that m is any integer such that $n = (m - 2) + (m + 3)$. If we simplify and use the previous expression for n, we obtain the equation $2k + 1 = 2m + 1$, and we can now use elementary algebra to conclude that $m = k$.

13. A less formal way of stating $P(x, y)$ is to say that the number m divides the number n (evenly, with no remainder). In these terms, here are the answers with explanations in some cases:
(a) FALSE. Certainly 4 does not divide 5.
(b) TRUE.
(c) FALSE. Some numbers do not divide others.
(d) TRUE. The number 1 evenly divides all numbers.
(e) FALSE. The first part gives a counterexample.
(f) TRUE. This follows from the comment in the third part.

14. We shall prove the contrapositive: *If x is rational, then so is x^3.* — The product of two rational numbers is rational, so x^2 is rational, and hence $x^3 = x^2 \cdot x$ must also be rational.

15. Consider the Pythagorean triples in the hint. We have $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$. Suppose we multiply the first equation by 13^2 and the second by 5^2. Then after doing some algebra we find that

$$39^2 + 52^2 = 65^2 = 25^2 + 60^2$$

and hence 65^2 is written as a sum of two squares in two different ways.

In the preceding example, the sum of the two squares is itself a perfect squares; if one is willing to take sums of two squares that are not necessarily perfect squares, then there are numerous smaller examples such as $5^2 + 5^2 = 50 = 7^2 + 1$ or $4^2 + 7^2 = 65 = 8^2 + 1^2$ or $6^2 + 7^2 = 85 = 9^2 + 2^2$.

16. The first few cubes are 1, 8 and 27; if we want to find a number that cannot be written as a sum of eight cubes, we might look for a number that is 7 more than some small multiple of 8. In fact, we cannot write 23 in the prescribed manner. Certainly this is impossible if we use all 1's or one 8, and if we use two 8's we also need seven 1's, and hence we need at least nine cubes to write 23. In fact, this turns out to be the smallest possible counterexample.

Note. A proof of Lagrange's theorem on expressing a positive integer as a sum of four (or fewer) perfect squares is given in Section 7.4 of the following book: I. N. Herstein, *Topics in Algebra* (2^{nd} Ed.), Wiley, New York, 1975, ISBN 0– 571–01090–1. — The proof is nominally at the advanced undergraduate level, but it might be more accurate to place it at the beginning graduate level.

17. In fact, 7 is not a sum of two squares and the cube of a nonnegative integer. Any such expression for a number less than 8 must be a sum of 4's and 1's, and at least four such numbers are needed to obtain a sum of 7. Once again, this is the smallest possible counterexample.

18. There are several sequences of steps that will achieve the stated goal, and we shall given the one in the supplement to Rosen. At each stage, let (a, b, c) denote the contents of the jugs holding 8, 5 and 3 gallons respectively. Then at the initial stage we have $(8, 0, 0)$. If we fill the 5 gallon jug using the 8 gallon jug, we get the configuration $(3, 5, 0)$. Now fill the 3 gallon jug using the 5 gallon jug to get the distribution $(3, 2, 3)$. Pour the contents of the 3 gallon jug back into the 8 gallon jug so that we have $(6, 2, 0)$, and pour the contents of the 5 gallon jug into the 3 gallon jug so that we have $(6, 0, 2)$. Next, fill the 5 gallon jug using the 8 gallon jug to obtain a distribution of $(1, 5, 2)$. Finally, top off the 3 gallon jug using the 5 gallon jug; this leaves us with $(1, 4, 3)$ and hence the 5 gallon jug now has 4 gallons of water and thus we have measured out 4 gallons of water as asked for in the problem.

Additional exercises to work

1. Suppose that P is the statement that x is the real number zero, Q is the statement that x is the real number one, and R is the statement that x is a real number. Then both $P \vee R$ and $Q \vee R$ are equivalent to R, but certainly P is not logically equivalent to Q.

Similarly, suppose P is the statement that the integer x is a perfect square, Q is the statement that the integer x is a perfect cube, and R is the statement that the intege x is a sixth power. Then both $P \wedge R$ and $Q \wedge R$ are logically equivalent to R, but P and Q are not logically equivalent because there are integers that are perfect squares but not perfect cubes and vice versa.■

2. The statement $\exists x \forall y Q(x, y)$ asserts there is an odd integer x such that for all odd integers x the number y^x is a perfect square. This is false. If x is odd then 3^x is never a perfect square. The statement $\forall y \exists x Q(x, y)$ asserts for every odd integer y there is an odd integer x such that y^x is a perfect square. This is also false for the same reasons.

Suppose that we look instead at the statements $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$. The first one is true; it suffices to take y to be a perfect square; if this is true then y^x will also be a perfect square. The second is true for the same reasons.■

Note. Here is a graphical explanation of why $\exists x \forall y Q(x, y)$ is true implies that $\forall y \exists x Q(x, y)$ is true. Set up a matrix whose rows correspond to the possibilities for x and whose columns correspond to the possibilities for y; it may have infinitely many rows or columns, but that need not concern us here. Insert a T or F in each entry depending upon whether $Q(x, y)$ is true or false. Then $\forall y \exists x Q(x, y)$ means that each column has a T somewhere, and $\exists x \forall y Q(x, y)$ states that one can always find a T in some fixed row (namely, the one corresponding to x).■

3. One way to work such a problem is to begin by listing all the integers between 1 and, say, 100. One then eliminates all the prime numbers, then all the numbers of the form $p + 1$, then all numbers of the form $p + 4$ and so on through $p + 81$, where the constants run through all perfect squares. One then checks to see which numbers have not been eliminated as potential counterexamples. The first one on the list is 25. This is a brute force approach but it works and really does not require all that much effort.■

4. Following the hint, write $n^3 - 1 = (n - 1)(n^2 + n + 1)$. If the number on the left hand side is prime, then one of the two factors on the right must be equal to 1. Since we are dealing with positive numbers, it follows that $n > 0$ and hence that the second factor is greater than 1. Therefore $n - 2 = 1$, so that $n = 2$, which means that $p = n^3 - 1 = 7$.■

5. If $3p + 1 = m^2$, then $3p = m^2 - 1 = (m - 1)(m + 1)$. By unique factorization into primes, one of the factors on the right must be equal to 3 and the other equal to p. If $3 = m + 1$ then $m = 2$ and $p = 2$; however, $7 = 3p + 1$ is not a perfect square in this case. On the other hand, if $3 = m - 1$, then $p = 5$ and $3p + 1 = 16 = 4^2$. Therefore the only possibility is $p = 5$.■

II.1 : Notation and first steps

Questions to answer

1. (*i*) We shall use the example of a deck of cards. Let A be the deck. Then the elements of A are single cards, and A is not a single card, so $A \notin A$.■

(*ii*) Suppose that A is a loaf of bread, so that the elements of A are slices of bread, and let B be a shipment containing loaves of bread, including A so that $A \in B$. Then $B \notin A$ because B is not a slice of bread.■

(*iii*) Let A be a slice of the loaf of bread B, and let B be one of the loaves in shipment C. Then $A \notin C$ because it is only a slice of bread and not an entire loaf.■

2. Once again let B be a loaf of bread in shipment C, and let A be some but not all of the slices of the loaf B. Only entire loaves are elements of C, so $A \notin C$.■

3. The appropriate interpretation of a line lying on a plane is that the subset given by the line is contained in the subset given by the plane.