SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 2

Fall 2006

III. Elementary constructions on sets (*continued*)

III.2 : Ordered pairs and products

Exercises to work

1. (1) Suppose that (x, y) lies in $A \times (B \cap D)$. Then $x \in A$ and $y \in B \cap D$. Since the latter means $y \in B$ and $y \in D$, this means that

$$(x, y) \in (A \times B) \cap (A \times D) \ .$$

Now suppose that (x, y) lies in the set displayed on the previous line. Since $(x, y) \in A \times B$ we have $x \in A$ and $y \in B$, and similarly since $(x, y) \in A \times D$ we have $x \in A$ and $y \in D$. Therefore we have $x \in A$ and $y \in B \cap D$, so that $(x, y) \in A \times (B \cap D)$. Thus every element of $A \times (B \cap D)$ is also a member of $(A \times B) \cap (A \times D)$ and vice versa, and therefore the two sets are equal.■

(2) Suppose that (x, y) lies in $A \times (B \cup D)$. Then $x \in A$ and $y \in B \cup D$. If $y \in B$ then $(x, y) \in A \times B$, and if $y \in D$ then $(x, y) \in A \times D$; in either case we have

$$(x, y) \in (A \times B) \cup (A \times D) \ .$$

Now suppose that (x, y) lies in the set displayed on the previous line. If $(x, y) \in A \times B$ then $x \in A$ and $y \in B$, while if $(x, y) \in A \times D$ then $x \in A$ and $y \in D$. In either case we have $x \in A$ and $y \in B \cup D$, so that $(x, y) \in A \times (B \cap D)$. Thus every member of $A \times (B \cup D)$ is also a member of $(A \times B) \cup (A \times D)$ and vice versa, and therefore the two sets are equal.■

(3) Suppose that (x, y) lies in $A \times (Y - D)$. Then $x \in A$ and $y \in Y - D$. Since $y \in Y$ we have $(x, y) \in A \times Y$, and since $y \notin D$ we have $(x, y) \notin A \times D$. Therefore we have

$$A \times (Y - D) \ \subset \ (A \times Y) - (A \times D) \ .$$

Suppose now that $(x, y) \in (A \times Y) - (A \times D)$. These imply that $x \in A$ and $y \in Y$ but $(x, y) \notin A \times D$; since $x \in A$ the latter can only be true if $y \notin D$. Therefore we have that $x \in A$ and $y \in Y - D$, so that

$$A \times (Y - D) \ \supset \ (A \times Y) - (A \times D) \ .$$

This proves that the two sets are equal.■

(4) Suppose that (x, y) lies in $(A \times B) \cap (C \times D)$. Then we have $x \in A$ and $y \in B$, and we also have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cap C$, while the second and fourth imply $y \in B \cap D$. Therefore $(x, y) \in (A \cap C) \times (B \cap D)$ so that

$$(A \times B) \cap (C \times D) \ \subset \ (A \cap C) \times (B \cap D) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the displayed equation. Then $x \in A \cap C$ and $y \in B \cap D$. Since $x \in A$ and $y \in B$ we have $(x, y) \in A \times B$, and likewise since $x \in C$ and $y \in D$ we have $(x, y) \in C \times D$, so that

$$(A \times B) \cap (C \times D) \ \supset \ (A \cap C) \times (B \cap D) \ .$$

Therefore the two sets under consideration are equal.■

(5) Suppose that (x, y) lies in $(A \times B) \cup (C \times D)$. Then either we have $x \in A$ and $y \in B$, or else we have $x \in C$ and $y \in D$. The first and third of these imply $x \in A \cup C$, while the second and fourth imply $y \in B \cup D$. Therefore (x, y) is a member of $(A \cup C) \times (B \cup D)$ so that

$$(A \times B) \cup (C \times D) \ \subset \ (A \cup C) \times (B \cup D) \ .$$

Supplementary note: To see that the sets are not necessarily equal, consider what happens if $A \cap C = B \cap D = \emptyset$ but all of the four sets A, B, C, D are nonempty. Try drawing a picture in the plane to visualize this.■

(6) Suppose that (x, y) lies in $(X \times Y) - (A \times B)$. Then $x \in X$ and $y \in Y$ but $(x, y) \notin A \times B$. The latter means that the statement

$$x \in A \ \textbf{and} \ y \in B$$

is false, which is logically equivalent to the statement

$$\textbf{either} \ x \notin A \ \textbf{or} \ y \notin B \ .$$

If $x \notin A$, then it follows that $(x, y) \in \big((X - A) \times Y\big)$, while if $y \notin B$ then it follows that $(x, y) \in \big(X \times (Y - B)\big)$. Therefore we have

$$\big(X \times Y\big) - \big(A \times B\big) \ \subset \ \big(X \times (Y - B)\big) \cup \big((X - A) \times Y\big) \ .$$

Suppose now that (x, y) lies in the set on the right hand side of the containment relation on the displayed line. Then we have $(x, y) \in X \times Y$ and also

$$\textbf{either} \ x \notin A \ \textbf{or} \ y \notin B \ .$$

The latter is logically equivalent to

$$x \in A \ \textbf{and} \ y \in B$$

and this in turn means that $(x, y) \notin A \times B$ and hence proves the reverse inclusion of sets.■

2. $A \times B$ consists of all ordered pairs (a, b) with $a \in A$ and $b \in B$. If there are no elements in either A or B, then there is no way to make an ordered pair of this type.■

3. If the intersection is empty, then it is impossible to construct ordered pairs of the form (x, y) with $x \in A$ and $y \in B$. We claim this means that $A \cap B = \emptyset$. If not and z belongs to both, then we would have (z, z) in the intersection of the Cartesian products.■

III.3 : Larger constructions

Exercises to work

1. The set $\$(F)$ is the set of all real numbers x such that $x \leq M$ for some positive real number M. Since $|x| + 1 > 0$ and $x < 1 + |x|$, it follows that every real number x belongs to a closed interval of the form $[-M, M]$ and hence The set $\$(F)$ contains all real numbers; on the other hand, since every element of the latter is a real number, the set in question is also contained in the real numbers, so it must be equal to the real numbers. — To characterize the intersection, notice that $0 \in [-M, M]$ for all M and hence 0 lies in the intersection. On the other hand, if $x \neq 0$ then $|x| > 0$ so that

$$x \quad \notin \quad [-\tfrac{1}{2}\,|x|, \tfrac{1}{2}\,|x|\,]$$

and hence x does not lie in the intersection. Thus the intersection is the set $\{0\}$.

The same reasoning shows that one obtains the identical answers for the union and intersection if the closed intervals $[-M, M]$ are replaced by the open intervals $(-M, M)$.■

2. We need to describe the sets $L(n)$ for all integers $n > 1$. If n is even, then this set is just the open interval $(0, 1)$ because even powers of 0 and 1 are equal to 0 and 1, even powers of negative numbers are positive, while if x is positive and $n > 1$ then $x^n < x$ if and only if $x < 1$. Thus the set $L(n)$ is equal to the open interval $(0, 1)$ if n is even. Suppose now that n is odd. The preceding discussion applies equally well if x is nonnegative. Furthermore, it $x < 0$ then we know that $x^n < x$ if $x < -1$, while $(-1)^n = -1$ and $0 > x^n > x$ if $-1 < x < 0$. Thus if n is odd then $L(n)$ is just the set $(-\infty, -1) \cup (0, 1)$.

It follows that the union of the sets $L(n)$ is equal to $(-\infty, -1) \cup (0, 1)$ and the intersection of the sets $L(n)$ is equal to $(0, 1)$.

3. Suppose that X is a subset of A for all $A \in C$; then $y \in X$ implies $y \in A$ for all such A so that $y \in \cap\{A \mid A \in C\}$, and hence we also have $X \subset \cap\{A \mid A \in C\}$. Conversely, if X is a subset of $\cap_C\, A$ then $X \in \cap_C\, P(A)$, and therefore we have the equality $\cap_A\, P(A) = P(\cap_C\, A)$.

Suppose now that X is a subset of A for somew $A \in C$; then it follows that X is a subset of $\cup_C\, A$, and this yields the second relationship in the exercise.

Finally, we have that $P(\{1\}) \cup P(\{2\})$ is a proper subset of $P(\{1, 2\})$ because $\{1, 2\}$ is not a subset of either $\{1\}$ or $\{2\}$.■

4. Yes, one can use $P(A) = P(B)$ to conclude that $A = B$. Define a subset of X to be *atomic* if it has no nonempty proper subsets. Then the atomic subsets are those which contain exactly one element. If $P(A) = P(B)$, then they have the same atomic subsets. Now for one point subsets we have that $\{x\} = \{y\}$ if and only if $x = y$, and hence $x \in A$ and $P(A) = P(B)$ imply $\{x\} \in P(B)$, so that $x \in B$. It follows that $A \subset B$, and reversing the roles of A and B we also obtain $B \subset A$ so that $A = B$.■

5. The "if" direction is trivial, so we focus on the "only if" direction here. Since $(a, b, c) = (u, c)$ where $u = (a, b)$ and $(x, y, z) = (v, z)$ where $v = (x, y)$, it follows that if the ordered triples are equal then $c = z$ and $u = v$, and the latter in turn implies that $a = x$ and $b = y$.■

6. With the given definition we have $\langle x, y, x\rangle = \langle x, y, y\rangle$ even if $x \neq y$.■

III.4 : A convenient assumption

Exercises to work

1. Follow the hint. Suppose that x has Russell type k, and consider the Russell type of $\{x\}$. Since x is the only element of $\{x\}$, it follows that any $\in$-sequence

$$a_n \;\in\; a_{n-1} \;\in\; \cdots \; a_1 \;\in\; \{x\}$$

must have $a_1 = x$. If x has Russell type k then there is a sequance of this sort where $n = k+1$ but there are no sequences of this type where $n \geq k+2$. Therefore there is an $\in$-sequence for $\{x\}$ which has $k+2$ terms but no sequences with more terms, and hence the Russell type of $\{x\}$ is $k+1$.

The preceding tells us if we have a set x with Russell type zero, we also have the set $\{x\}$ with Russell type one, and likewise the singleton for the latter has Russell type two, and so on. Therefore the proof reduces to verifying that there is a set of Russell type zero, and the empty set satisfies this condition.■

2. If a set S has Russell type k, then every element of S will have Russell type at most $k-1$, and conversely if every element of S has Russell type at most $k-1$, then S has Russell type at most k.

Suppose now that A and B have finite Russell types p and q respectively and that r is the larger of p and q. Then the Russell type of $A \cup B$ is less than or equal to r.■

3. The end of any $\in$-sequence for $A \times B$ must have terms of the form

$$\cdots \ \{\ \{a\},\ \{a,b\}\ \} \ = \ (a,b) \ \in \ A \times B$$

and if the sequence continues then the next term down must be $\{a\}$ or $\{a,b\}$, while the term after that must be a or b. If A anad B have finite Russell type, then there is some k such that every $\in$-sequence ending with an a or a b must have at most $k+1$ terms. By the discussion above, it follows that every $\in$-sequence ending with $A \times B$ must then have at most $k+3$ terms.■

4. Every $\in$-sequence ending in $P(A)$ must end with terms of the form $b \in B \in P(A)$, where $B \subset A$. Thus if A has Russell type n, then $P(A)$ has Russell type $n+1$.■

IV. Relations and functions

IV.1 : Binary relations

Exercises to work

GENERAL REMARK. There are several exercises which ask whether a given binary relation is reflexive, symmetric, antisymmetric or transitive. We shall only work out a few representative examples in detail and give yes/no answers for the others. Details for the remaining examples appear in the handbooks written to accompany Rosen's text.

1. (a) This relation is not reflexive because $(1,1)$ and $(4,4)$ are not elements of the subset. It is not symmetric because it contains $(2,4)$ but not $(4,2)$. It is not antisymmetric because it contains $(2,3)$ and $(3,2)$, and of course $2 \neq 3$. To see it is transitive, one needs to enumerate all the pairs of ordered pairs (a,b) and (b,c) in the relation:

[1] $(2,2)$, $(2,2)$
[2] $(2,2)$, $(2,3)$
[3] $(2,2)$, $(2,4)$
[4] $(2,3)$, $(3,2)$
[5] $(2,3)$, $(3,3)$
[6] $(2,3)$, $(3,4)$

[7] $(3,2)$, $(2,2)$
[8] $(3,2)$, $(2,3)$
[9] $(3,2)$, $(2,4)$
[10] $(3,3)$, $(3,3)$
[11] $(3,3)$, $(3,4)$

The transitivity of this relation amounts to saying that for each of these cases the corresponding ordered pair (a,c) lies in the relation. One checks this out on a case by case basis.■

(b) This relation is reflexive, symmetric and transitive but not antisymmetric. We shall only give details for the first two because the others are worked in a manner similar to the previous exercise. The relation is reflexive because it contains each ordered pair (x,x). To show it is symmetric, one must list all the ordered pairs in the relation

$$(1,1),\ (2,2),\ (2,1),\ (1,2),\ (3,3),\ (4,4)$$

and check that the pairs with the entries switched

$$(1,1),\ (2,2),\ (1,2),\ (2,1),\ (3,3),\ (4,4)$$

also belong to the relation, which is straightforward.■

(c) This relation is symmetric, but not reflexive, antisymmetric or transitive. We have already done examples for the first three types, so we shall only give details for the last conclusion. This follows because the relation contains $(2,4)$ and $(4,2)$ but neither $(2,2)$ nor $(4,4)$. If a relation is transitive and contains both $(2,4)$ and $(4,2)$, then it must also contain both $(2,2)$ and $(4,4)$.■

(d) This relation is not reflexive, symmetric or transitive, but it is antisymmetric. The latter is **vacuously true** because the relation does not contain two ordered pairs of the form (a,b) and (b,a)!■

(e) This relation is not reflexive, symmetric, antisymmetric or transitive. However, the statement of this problem differs from the corresponding statement in Rosen; specifically, the latter asks about the relation given by $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$. The relation in Rosen's original problem is reflexive, symmetric, antisymmetric and transitive. We shall only discuss the antisymmetry property (problems of the other types have already been considered). The only pairs (a,b) such that both (a,b) and (b,a) are in the relation are the diagonal pairs of the form (a,a), so if one has aRb and bRa then $a=b$ follows.■

(f) This relation is not reflexive, symmetric, antisymmetric or transitive.■

2. (c) This relation is reflexive, symmetric and transitive, but not antisymmetric.■

(d) This relation is not reflexive, not symmetric and not transitive, but it is antisymmetric. We shall only check the last of these. If xRy and yRx then we have $x=2y$ and $y=2x$. The only way this can happen is if $x=y=0$, and of course it does happen in this case.■

(e) This relation is reflexive and symmetric, but it is neither antisymmetric nor transitive.■

(f) This relation is symmetric, but it is not reflexive, not antisymmetric and not transitive.■

3. (a) This relation is symmetric, but it is not reflexive, not antisymmetric and not transitive.■

(b) This relation is symmetric, but it is not reflexive, not antisymmetric and not transitive.■

(c) This relation is symmetric, but it is not reflexive, not antisymmetric and not transitive.■

(g) This relation is not reflexive, not symmetric and not transitive, but it is antisymmetric. We shall only check the last of these. If xRy and yRx then we have $x = y^2$ and $y = x^2$. Substituting the first equation into the second, we obtain $y = y^4$, so that either $y = 0$ or $y^3 = 1$. If $y = 0$ then we must also have $x = 0$. If $y^3 = 1$ and we are dealing with real numbers, then we must have $y = 1$, which in turn implies $x = 1$. [*Note:* If we allowed complex numbers then the relation would not be antisymmetric, for if x is a non-real cube root of 1 and $y = x^2$ then $y \neq x$ but $x = y^2$.]■

(h) This relation is not reflexive, not symmetric, not transitive, and not antisymmetric. We shall only check the last two of these. A counterexample to transitivity is given by $(x, y) = (\frac{1}{3}, \frac{1}{2})$ and $(y, z) = (\frac{1}{2}, \frac{3}{5})$. For these choices we have $x > y^2$ and $y > z^2$ but $x < z^2$. A counterexample to antisymmetry is given by the same (x, y).■

4. Yes. For the reflexive property, if for each x we have $(x, x) \in R_1$ and $(x, x) \in R_2$, then we also have $(x, x) \in R_1 \cap R_2 \subset R_1 \cup R_2$. For the symmetry property, suppose first that $(x, y) \in R_1 \cap R_2$. Then $(x, y) \in R_1$ and $(x, y) \in R_2$, and since R_1 and R_2 are symmetric it follows that $(y, x) \in R_1$ and $(y, x) \in R_2$, so that $(y, x) \in R_1 \cap R_2$. Suppose now that that $(x, y) \in R_1 \cup R_2$. Then $(x, y) \in R_1$ or $(x, y) \in R_2$, and since R_1 and R_2 are symmetric it follows that $(y, x) \in R_1$ or $(y, x) \in R_2$ respectively, so that $(y, x) \in R_1 \cap R_2$.■

5. (a) Equivalence relation.■

(b) Not reflexive and not transitive.■

(c) Equivalence relation.■

(d) Not transitive.■

(e) Not symmetric and not transitive.■

6. (a) Equivalence relation.■

(b) Equivalence relation.■

(c) Not transitive.■

(d) Not transitive.■

(e) Not transitive.■

7. The relation S is reflexive, for R is reflexive and xRx and xRx imply xSx. Suppose now that xSy. Then xRy and yRx, and by definition this also implies ySx. Finally, suppose that xSy and ySz. Then we have xRy and yRx and, we also have yRz and zRy. By the transitivity of R these imply that xRz and zRx, which means that xSz. Therefore S is an equivalence relation.■

8. We first prove the ($\Longrightarrow$) implication. Suppose that R is an equivalence relation. Then it is automatically reflexive. Suppose now that xRy and yRz. Then we also have xRz because R is transitive. But since R is symmetric the latter implies zRx and hence T is circular. Now we prove the ($\Longleftarrow$) implication. By assumption R is reflexive. To show that it is symmetric, suppose that xRy. If we combine this with xRx (since R is reflexive) and the circular property we conclude that yRx. Finally, if xRy and yRz, then zRx since R is circular. However, we have shown that R is symmetric, and therefore we also have xRz so that R is transitive. Hence R is an equivalence relation.■

9. We have $(x, y)P(x, y)$ because $xy = xy$, and if $(x, y)P(z, w)$ then $xw = yz$, which is equivalent to $zy = wx$, which means that $(z, w)P(x, y)$. Finally, if $(x, y)P(z, w)$ and $(z, w)P(u, v)$, then $xw = yz$ and $zv = uw$. Multiplying these equations together yields $xwzv = yzuw$, and since $w \neq 0$ it follows that $xzv = yzu$. If $z \neq 0$ then we may divide both sides of the equation by z and obtain $xv = yu$, which implies $(x, y)P(u, v)$.

Suppose now that $z = 0$. Then $0 = yz = xw$ and since $w \neq 0$ it follows that $x = 0$. Likewise, $0 = zv = uw$ implies $u = 0$. But then we have $xv = 0v = 0 = 0w = uw$, so that $(x, y)P(u, v)$ in this case too. Therefore we have shown that the relation is an equivalence relation.

To prove the final assertion, we first show there is at least one r such that $(x, y) = (r, 1)$. Specifically, if $r = x/y$, then $yr = x = x \cdot 1$. Next, we must show there is only one such r, so suppose we have $(x, y)P(s, 1)$. Then by the definition of the equivalence relation we have $ys = x$, so that $s = x/y$, and hence s must be equal to the value of r give previously.■

10. Taking logarithms, we find that $(x, y)Q(z, w)$ if and only if $w \ln x = y \ln z$. One can now proceed as in the previous exercise to show that Q is reflexive and symmetric, and also that Q is transitive with separate consideration for the cases $\ln z \neq 0$ and $\ln z = 0$. Therefore the relation Q is also an equivalence relation.■

11. (i) There is a figure to illustrate the argument in the file `knightmoves.JPG`, with the knight starting at its usual position at time 1 and different colors indicating new possibilities for its positions at times 2 through 6. The picture indicates that the knight can reach every square in 6 moves or less. Writing everything down in detail is left to the reader.■

(ii) The figure `knight2.JPG` shows how a knight starting at $(0, 0)$ can reach the diagonally adjacent square $(-1, -1)$ after two moves and the horizontally adjacent square $(1, 0)$ after three moves, with the first move going to $(1, 2)$. Symmetry considerations yield all the eight cases as follows:

$$(0,0) \to (1,2) \to (-1,1) \to (1,0)$$
$$(0,0) \to (1,-2) \to (-1,-1) \to (1,0)$$
$$(0,0) \to (-1,-2) \to (1,1) \to (-1,0)$$
$$(0,0) \to (-1,2) \to (1,-1) \to (-1,0)$$
$$(0,0) \to (2,1) \to (1,-1) \to (0,1)$$
$$(0,0) \to (-2,1) \to (-1,-1) \to (0,1)$$
$$(0,0) \to (-2,-1) \to (1,1) \to (0,-1)$$
$$(0,0) \to (2,-1) \to (-1,1) \to (0,-1)$$

Note that there is duplication in this list, for each adjacent square appears in exactly two sequences.■

12. The union of the relations is that a is a multiple of b or b is a multiple of a, and the intersection is that a is a multiple of b and b is a multiple of a. The first of these cannot be simplified, but the second can as follows: if $a = xb$ and $b = ya$, then $a = xb = xya$ implies that $xy = 1$, so that $x = y = \pm 1$, and hence the intersection is the relation that $a = \pm b$.■

13. As noted in the hint, the statement $x\ S \circ T\ y$ means that $y - 2 = b(ax - 1)$ where a and b are ± 1. There are two sign choices for each of a and b, and since they may vary independently there are a total of four possible values of y related to a given value of x under the relation $S \circ T$:

$$y - 2 = 2 + (x - 1) = x + 1$$
$$y - 2 = 2 + (-x - 1) = 1 - x$$
$$y - 2 = 2 - (x - 1) = 3 - x$$

$$y - 2 = 2 - (-x - 1) = 3 + x$$

If $x = 1$ we obtain the possible values of $2, 0, 4$ for y; note that the first and third formulas give the same value for y. On the other hand, If $x = 2$ we obtain the possible values of $3, -1, 1, 5$ for y.∎

14. Suppose that $x\ S\circ(T_1 \cup T_2)\ y$. Then we have $x\ S\ z$ and $z\ (T_1 \cup T_2)\ y$ for some z. If $z\ T_1\ y$ then we have $x\ S\circ T_1\ y$ and on the other hand if $z\ T_2\ y$ then we have $x\ S\circ T_2\ y$; in both cases we have

$$x\ [S\circ T_1\ \cup\ S\circ T_2]\ y$$

and therefore we have $S\circ(T_1 \cup T_2) \subset S\circ T_1 \cup S\circ T_2$. — Conversely, suppose that $x\ [S\circ T_1\ \cup\ S\circ T_2]\ y$. Then either there is some z such that $x\ S\ z$ and $z\ T_1\ y$ or else there is some z such that $x\ S\ z$ and $z\ T_2\ y$. In both cases we have $z\ (T_1 \cup T_2)\ y$ and hence $x\ S\circ(T_1 \cup T_2)\ y$.

Suppose now that $x\ S\circ(T_1 \cap T_2)\ y$. Then we have $x\ S\ z$ and $z\ (T_1 \cap T_2)\ y$ for some z. In particular, we have $z\ T_1\ y$ and $z\ T_2\ y$, which means that

$$x\ [S\circ T_1\ \cap\ S\circ T_2]\ y$$

and consequently $S\circ(T_1 \cap T_2) \subset S\circ T_1 \cap S\circ T_2)$.

Here is an example where the containment in the preceding paragraph is proper. Take a set consisting of the four objects $\{x, a, b, y\}$, and define binary relations S, T_1, T_2 such that the following are the only true statements about them:

$$x\ S\ a, \qquad x\ S\ b, \qquad a\ T_1\ y, \qquad b\ T_2\ y$$

Then $T_1 \cap T_2 = \emptyset$, and hence we must also have $S\circ(T_1 \cap T_2) = \emptyset$; in other words, there are no choices of u and v for which $u\ S\circ(T_1 \cap T_2)\ v$ is true. On the other hand, by construction we know that $x\ S\circ T_1\ y$ and $x\ S\circ T_2\ y$ are both true, so that

$$x\ [S\circ T_1\ \cap\ S\circ T_2]\ y$$

is true, Therefore the binary relations $S\circ T_1\ \cap\ S\circ T_2$ and $S\circ(T_1 \cap T_2)$ are not equal.∎