SOLUTIONS TO EXERCISES FOR

MATHEMATICS 144 — Part 3

Fall 2006

IV. Relations and functions

IV.2 : Partial and linear orderings

Exercises to work

1. We shall prove the statement about intersections of partial orderings first. By definition we have $x\ (P_1 \cap P_2)\ y$ if and only if $x\ P_1\ y$ and $x\ P_2\ y$. REFLEXIVE PROPERTY. Since $x\ P_1\ x$ and $x\ P_2\ x$, we have $x\ (P_1 \cap P_2)\ x$. SYMMETRIC PROPERTY. In this case we are given $x\ P_1\ y$, $x\ P_2\ y$, $y\ P_1\ x$ and $y\ P_2\ x$. Since P_1 and P_2 are both partial orderings it follows that $x = y$. TRANSITIVE PROPERTY. We are now given $x\ P_1\ y$, $x\ P_2\ y$, $y\ P_1\ z$ and $y\ P_2\ z$. Since P_1 and P_2 are both partial orderings it follows that $x\ P_1\ z$ and $x\ P_2\ z$, so that $x\ (P_1 \cap P_2)\ z$.

Finally, we need to show that $P_1 \cup P_2$ is not a partial ordering. Take a set with two elements a and b, and let P_1 and P_2 be the unique partial orderings for which $a\ P_1\ b$ and $b\ P_2\ a$. Then $a \neq b$ but $a\ (P_1 \cup P_2)\ b$ and $b\ (P_1 \cup P_2)\ a$, so the union relation is not antisymmetric and hence it cannot be a partial ordering.∎

Further question. If the two orderings in the previous problem are linear, then the intersection is not necessarily a linear ordering (look at the second example; in this case a and b are not comparable).∎

2. The elements greater than $(2, 3)$ in the lexicographic ordering are $(2, 4)$, $(3, n)$ and $(4, n)$ where $n = 1, 2, 3, 4$. Likewise, the elements less than $(3, 1)$ are $(1, n)$ and $(2, n)$ for the same range of values for n.∎

3. We shall work the parts in order.

(1) The reflexive property follows from the construction. To prove the symmetric property, suppose that $[a, b]\ P\ [c, d]$ and $[c, d]\ P\ [a, b]$. If $[a, b] \neq [c, d]$, then by definition we have $b < c$ and $d < a$. Since $a < b$ and $c < d$, this yields the impossible chain of strict inequalities $a < b < c < d < a < b$. Thus the only logical possibility is for the two intervals to be equal. Finally, suppose we have $[a, b]\ P\ [c, d]$ and $[c, d]\ P\ [e, f]$. The conclusion is trivial if either $[a, b] = [c, d]$ or $[c, d] = [e, f]$, so let us assume that neither holds. We then have $b < c < d < e < f$, and this implies $[a, b]\ P\ [e, f]$.

(2) The statement of the exercise is equivalent to the statement that if $[a, b]$ and $[c, d]$ are comparable but unequal, then they must be disjoint. Suppose the two intervals are unequal. Then if $[a, b]\ P\ [c, d]$, we have $b < c$ which means that the two intervals are disjoint, while if $[c, d]\ P\ [a, b]$, we have $d < a$ which also means that the two intervals are disjoint.

(3) Consider the intervals $[0, 1]$ and $[0, 2]$. These intervals are not equal and not disjoint, so by the preceding part of the exercise they cannot be comparable with respect to the partial ordering P.∎

4. If we have a linearly ordered chain $S_1 < \cdots < S_m$ then the number of elements in S_k is at least one more than the number of elements in S_{k-1}. Since S_1 contains at least zero elements, this means that the number of elements in S_m is at least $m-1$. Since S has n elements, this means that $m-1 \leq n$ or $m < n+2$. To get a linearly ordered set with $n+1$ elements, start with $S_1 = \emptyset$, and for $1 < k \leq n+1$ let $S_k = \{1, \cdots, k-1\}$.■

5. The relation is reflexive because $p(x) \leq p(x)$ for all x. It is antisymmetric because $p(x) \leq q(x)$ for all x and $q(x) \leq p(x)$ for all x imply $p(x) = q(x)$ for all x, and hence $p = q$. It is transitive because $p(x) \leq q(x)$ for all x and $q(x) \leq r(x)$ for all x imply $p(x) \leq r(x)$ for all x, so that $p \leq r$.

To show this partial ordering is not a linear ordering we need to find two polynomials p and q such that p is not less than or equal to q and vice versa. It will be enough to find p and q together with real numbers a and b such that $p(a) > q(a)$ but $p(b) < q(b)$, for the first implies $p \leq q$ is false and the second implies that $p \geq q$ is false.

Specifically, take p to be the constant polynomial with value 1 and let $q(x) = x$. Then we have $q(1) > p(1)$ but $q(-1) < p(-1)$.■

6. (a) ℓ and m are the maximal elements.■

(b) a, b and c are the minimal elements.■

(c) No. There is no element that is greater than or equal to both ℓ and m.■

(d) There is no least element, because there is no element that is less than or equal to all of a, b, c.■

(e) ℓ and m are the upper bounds.■

(f) There is none; such an element would have to be less than or equal to both of the upper bounds described above, and no such upper bound exists.■

(g) There are no elements that are less than or equal to all three of f, g, h.■

(h) Since there is no lower bound, there cannot be a greatest lower bound.■

15. There are two things two prove, the first being the general fact about betweenness and the second being the assertion that exactly one element lies between the other two. To prove the first, note that y is between x and z means that $x < y < z$ or $z < y < x$, and this is equivalent to saying that $z < y < x$ or $x < y < y$, which is the condition for y to be between z and x.

There are six different ways of permuting the variables x, y, z in the statement, "x is between y and z." By the previous paragraph they can be grouped into three pairs such that the two statements in each pair are logically equivalent:

(1) "y is between x and z" and "y is between z and x."
(2) "z is between x and y" and "z is between y and x."
(3) "x is between y and z" and "x is between z and y."

Given an arbitrary subset of three distinct elements, we need to prove that one of these pairs of statements will be true and the others will be false. The discussion splits into several cases.

Case 1. Suppose that $x < y$. Then there are subcases depending upon z is related to x and y. SUBCASE 1A. $y < z$. — In this case y is between x and z. SUBCASE 1B. $z < y$ and $x < z$. — In this case z is between x and y. SUBCASE 1C. $z < y$ and $z < x$. — In this case x is between z and y.

Case 2. Suppose that $y < x$. There are again subcases depending upon z is related to x and y. SUBCASE 2A. $z < y$. — In this case y is between x and z. SUBCASE 2B. $y < z$ and $z < x$. — In this case z is between x and y. SUBCASE 2C. $y < z$ and $x < z$. — In this case x is between z and y.

In every case we have shown that one element is between the other two. We shall conclude by showing that if y is between x and z, then z is not between y and x and x is not between y and z. If we can show this, we are essentially done, for the other two cases will follow by interchanging the roles of x, y and z in the argument.

Suppose first that y is between x and z and z is also between y and x. Then we have $x < y < z$ or else we have $z < y < x$. Similarly, we also have $y < z < x$ or $x < z < y$. There are four possible pairs of inequalities under the given assumptions. We shall show they all lead to contradictions.

(1) $x < y < z$ and $y < z < x$ lead to the conclusion $x < x$, which we know is false.

(2) $x < y < z$ and $x < z < y$ are inconsistent because only one of the relations $y < z$ and $z < y$ can be true.

(3) $z < y < x$ and $y < z < x$ again lead to the conclusion $x < x$, which we know is false.

(4) $z < y < x$ and $x < z < y$ are again inconsistent because only one of the relations $y < z$ and $z < y$ can be true.

Therefore we have shown that if y is between x and z, then neither x nor z can be between the remaining two points, and as noted before this suffices to complete the proof.■

IV.3 : Functions

Exercises to work

1. There are two reasons why it is not the graph of a function. The first is that Grover Cleveland served nonconsecutive terms and was succeeded by Benjamin Harrison after the first term and by William McKinley after the second. The other reason is that the successor to the current President is not presently known,■

2. The main point is to find the graph. If A is nonempty, then we simply take its graph to be $A \times \{x\}$; this represents the constant function whose value is x at every element of A. If A is empty, then we take the graph to be the empty set. These prove existence of a function from A to $\{x\}$.

To prove the function is unique, if A is empty, then its graph is a subset of $A \times \emptyset = \emptyset$ and thus is equal to the graph of the previously defined example. If A is not empty, then $A \times \{x\}$ is the only subset of itself that contains a point with first coordinate a for all choices of $a \in A$, and therefore the graph of an arbitrary function from A to $\{x\}$ must be equal to that of the example in the previous paragraph.■

3. It is vacuously true that $\emptyset = \emptyset \times X$ is the graph of a function from $\emptyset$ to X, and conversely since every subset of $\emptyset \times X$ is empty it follows that there is only one possibility for the graph.

On the other hand, if X is not empty then $X \times \emptyset = \emptyset$, and hence every subset of $X \times \emptyset$ is also empty; therefore if $x \in X$ is arbitrary then there is no ordered pair of the form (x, y) in a subset of $X \times \emptyset$, so no subset of the latter can be the graph of a function.■

4. The domain is the set of all positive integers, and the range is the set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \ .\blacksquare$$

5. The domain is the set of all positive integers, and the range is the set of integers

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8\} \ .$$

Note that at least one of the digits from 1 through 9 must appear, and any higher number can also appear.■

6. (*i*) We need to solve the equation $f(x) = 3$, where $f(x) = 3x - 7$. But if $3x - 7 = 3$, then $x = \frac{10}{3}$, and hence the inverse image is $\{\frac{10}{3}\}$.■

(*ii*) We need to find the singleton set whose element is $f(5) = 3 \cdot 5 - 7 = 8$. Therefore the set we want is $\{8\}$.■

(*iii*) We need to find all x such that $-7 \leq 3x - 7 \leq 2$. Adding 7 to everything in sight we see that the inequalities are equivalent to $0 \leq 3x \leq 9$, which in turn is equivalent to $x \in [0, 3]$. Hence in this case the inverse image is equal to $[0, 3]$.■

(*iv*) We need to find the set of all y such that $y = 3x - 7$ for $2 \leq x \leq 6$. This turns out to be the interval $[-1, 111$, which is thus the image of the given interval.■

(*v*) The image of the empty set is always equal to the empty set.■

(*vi*) We need to find all x such that $3 \leq 3x - 7 \leq 5$ or $3x - 7 \leq 8 \leq 10$, and these are equivalent to $10 \leq 3x \leq 12$ or $15 \leq 3x \leq 17$. Thus in this case the inverse image is equal to $[\frac{10}{3}, 4] \cup [5, \frac{17}{3}]$.■

7. (*i*) This is just the singleton set containing $f(-1) = 0$.■

(*ii*) The inequalities $0 \leq (x + 1)^2 \leq 1$ hold if and only if $-1 \leq x + 1 \leq 1$, which in turn is equivalent to $x \in [-2, 0]$, so $[-2, 0]$ is the inverse image of the set in the problem.■

(*iii*) Since $(x+1)^2 \geq 0$ for all x, the inverse images of $[-1, 1]$ and $[0, 1]$ are equal, so the answer is the same as for the previous problem.■

(*iv*) The inequalities $-3 \leq (x + 1)^2 \leq 5$ are equivalent to $0 \leq (x + 1)^2 \leq 5$, which in turn is equivalent to $-5 \leq x + 1 \leq 5$, so that the inverse image in this case is equal to $[-6, 4]$.■

(*v*) The inverse image of $[-3, -1]$ is empty because f takes nonnegative values. As in (*v*) from the previous exercise, this implies that the set to be determined is the empty set.■

(*vi*) By (*iii*) we know that the inverse image of $[-1, 1]$ is equal to $[-2, 0]$, and the image of the latter under f is equal to $[0, 1]$, because $x \in [-2, 0]$ implies $x+1 \in [-1, 1]$, so that $0 \leq (x+1)^2 \leq 1$.■